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DISTRIBUTION, APPARENT MASS, AND MOMENT  
OF INERTIA OF A DISK PENDULUM  
OSCILLATING AT LOW FREQUENCY H.S.  
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Theoretical Pressure Distribution, Apparent  
Mass, and Moment of Inertia of a Disk Pendulum  
Oscillating at Low Frequency

By

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of the requirements for the degree of Master of Science

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# **ABSTRACT**

Equations are developed which give the pressure profile, the forces and torques on a disk pendulum by means of point source wave theory from acoustics. The pressure, force and torque equations for an un baffled disk are developed. These equations are then used to calculate the apparent mass and apparent inertia for the pendulum.

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## SYMBOLS

$a$	radius of disk
$b$	distance from center line to point of pressure application
$C_D$	drag coefficient
$c$	speed of sound in air (at any pressure)
$E(\frac{\pi}{2}, m)$	complete elliptic integral of the second kind
$F$	force on the pendulum surface
$G$	Green's function
$g$	acceleration due to gravity
$I$	total moment of inertia (structural + apparent inertia)
$j$	imaginary, $\sqrt{-1}$
$K(\frac{\pi}{2}, m)$	complete elliptic integral of the first kind
$k$	wave number $\frac{\omega}{c}$
$l_{cg}$	distance from pivot point to center of disk
$l_{aero}$	distance from pivot point to center of aerodynamic pressure
$m$	mass
$P$	period
$p$	atmospheric pressure
$p_i$	pressure at differential element area $i$ on the disk due to all other sources on the surface of the disk
$p'_s$	pressure at a differential element of area $s'$ due to a source at $s$



$r$	dummy radial variable for integrating pressure due to other sources on a disk
$r_{\max}$	value of $r$ for source at edge of disk
$S$	reference area
$s$	differential area of source of pressure
$s'$	differential area of application of pressure
$t$	time
$\hat{u}$	maximum speed of a sinusoidally oscillating disk
$V$	velocity of pendulum c.g.
$v$	velocity
$x_m$	upper limits of integration on $r$
$Z$	impedance
$z$	normal to boundary
$\theta$	angle of swing of pendulum
$\lambda$	wavelength
$\rho_0$	ambient air density
$\sigma$	radial pressure coordinate from center of disk
$\tau$	torque
$\phi$	dummy angular variable for integrating pressure due to other sources
$\psi$	dummy angular variable for integrating force or torque due to other sources
$\omega$	angular rate of oscillator or pendulum

x

Subscripts:

a	apparent
cg	center of gravity
i	incident at a differential area
s	area of pressure source
s'	area where pressure is applied

Dots over symbols denote differentiation with respect to time.

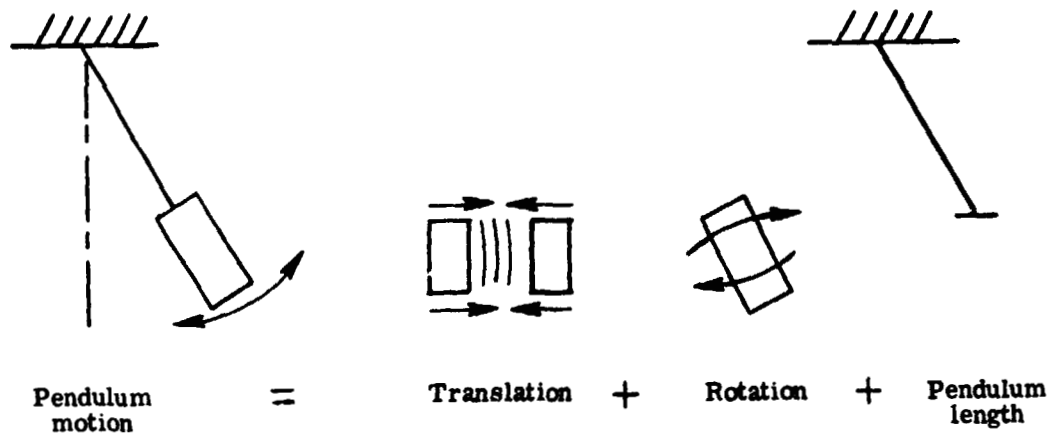
Hats over symbols denote maximum values. Roman numeral subscripts I and II denote pressures due to translation and rotation, respectively. Arabic subscripts denote successive terms in the total pressure.

## INTRODUCTION

An expression for the total pressure on the surface of a rigid circular disk vibrating in a baffle was worked out by Rayleigh (reference 1, Vol. II page 162). However, Rayleigh did not give an expression for the pressure distribution across the disk. This was done subsequently by McLachlen (reference 2, p. 1012) by means of an expansion using hypergeometric functions. Such a pressure distribution is necessary if correct moments are to be calculated for a disk in rotation about a diameter as frequently occurs in experimental work.

The work of these authors can be extended readily, in the special case of low frequencies, to include the pressure on a disk pendulum. The pendulum can be considered as an unbaffled disk in rotation and translation. The net affect of the absence of a baffle is to reduce the pressure on both faces since air is free to travel around the disk to the opposite face as the pendulum swings.

The pressure acting on the face of a disk represents an inertial force if dissipative effects are neglected. Hence the pendulum in motion appears more massive than when not in motion. The result is that the period of the pendulum will be different at different air densities. Physically the pendulum drags a certain amount of air with it as it swings. This air is referred to as attached mass, entrained mass, apparent mass, or virtual mass by various authors. For a pendulum, rotational effects must also be considered; so apparent inertias are also present as shown in sketch (a).



Sketch (a)

It will be the objective of this paper to show that the theoretical acoustic pressure profile can be investigated by breaking the equations of motion into two special cases: (a) a pure translation and (b) a pure rotation about a center line of the disk. A combination of these effects, along with consideration of the pendulum support distance, affords a description of the radiation and attached mass by means of the principle by superposition and hence represents the total effect.

A comparison will be made of these results with the theoretical results obtained from classical hydrodynamics and from experiment.

### Theoretical Model

In the present study two models can be considered as technically feasible. One of these consists of ignoring the absence of a baffle and thus treating the pendulum as a distributed monopole sound source at low frequency. In this case the Green's function becomes the free space Green's function for outgoing waves with infinite boundary conditions in the positive half space. Under these conditions the pressure is finite at the edge of the disk. The second model consists of assuming that the pendulum is a distributed acoustic dipole. This leads to a somewhat more complicated model than a monopole disk since under these conditions a term must be included in the Green's function for pressure which serves to force the free space term to zero at the boundary. By symmetry it also follows that the pressure must be zero at all points in the plane of the disk beyond the boundary.

In the present paper a dipole model will be used to calculate the attached mass of the disk.

The present paper is similar in some respects to a number of papers already in the literature. However, several of significant differences are discernable. M. Strassberg (reference 3, p. 520) considered the radiation field only at large distances from the oscillating body. The result is that the force may be computed correctly but no detailed description of the pressure distribution is possible, although an approximate value can be obtained by setting  $F = ma$ . Bouwkomp (reference 4) and Wiener (reference 5) treated the sound fields diffracted around various obstacles. As such their

papers are valuable in that they complement the present work. However, in both papers, fixed integration limits result in a more complicated summation formulation which must satisfy matching conditions at the boundary and, as such, these papers are more like perturbation problems than is the present paper. Crane (reference 6) takes an approach which is closest to that of the present work in that a free space potential field is considered along with an auxiliary field to satisfy the boundary conditions. However, crane assumes zero potential from the edge of the disk to infinity whereas in the present paper this condition arises out of symmetry considerations and the pressure is guaranteed zero only at the boundary. This results in Crane following Bouwkomp's paper by insisting on a fixed coordinate center for integration. The result is that in references 4, 5 and 6 it becomes necessary to describe the pressure from a coordinate system which is centered at the center of the disk; and hence these authors are forced to use summations of Bessel and Legendre functions. In the present paper the pressure is calculated in a coordinate system located in each instance at the assumed receptor point. While circular symmetry is sacrificed by this approach the algebra becomes such that the only expansions required are those of the ordinary exponential. It should be noted that in none of the above references is rotational motion about a disk diameter considered, however, Mangulis (reference 7) does consider this for a baffled disk.

## MATHEMATICAL MODEL

### Monopole Disk Theory

Consider, first, a rigid disk vibrating in an infinite baffle with simple harmonic motion. The total force is determined by integrating the pressure over the surface of the disk. The pressure at some infinitesimal area  $ds'$  produced by the oscillation of some nearby area  $ds$  is given by equation (1)

$$dp_{s'} = \frac{j \rho_0 c k \hat{u} e^{j(\omega t - kr)}}{4\pi r} ds \quad (1)$$

where  $r$  is the distance between  $ds$  and  $ds'$ ,  $\hat{u}$  is the maximum speed of the oscillator,  $k$  the wave number,  $c$  the speed of sound,  $\rho_0$  the ambient air density, and  $S$  the surface area of the disk. Thus the total pressure acting on one face can be obtained by integrating over the surface of the disk treating each infinitesimal area  $ds'$  as a point source (reference 8, p. 178).

The total reaction force acting on the disk is the integral of the pressure over the surface.

$$F = - \iint_S p_{s'} ds' \quad (2)$$

Substituting equation (1) into equation (2) yields

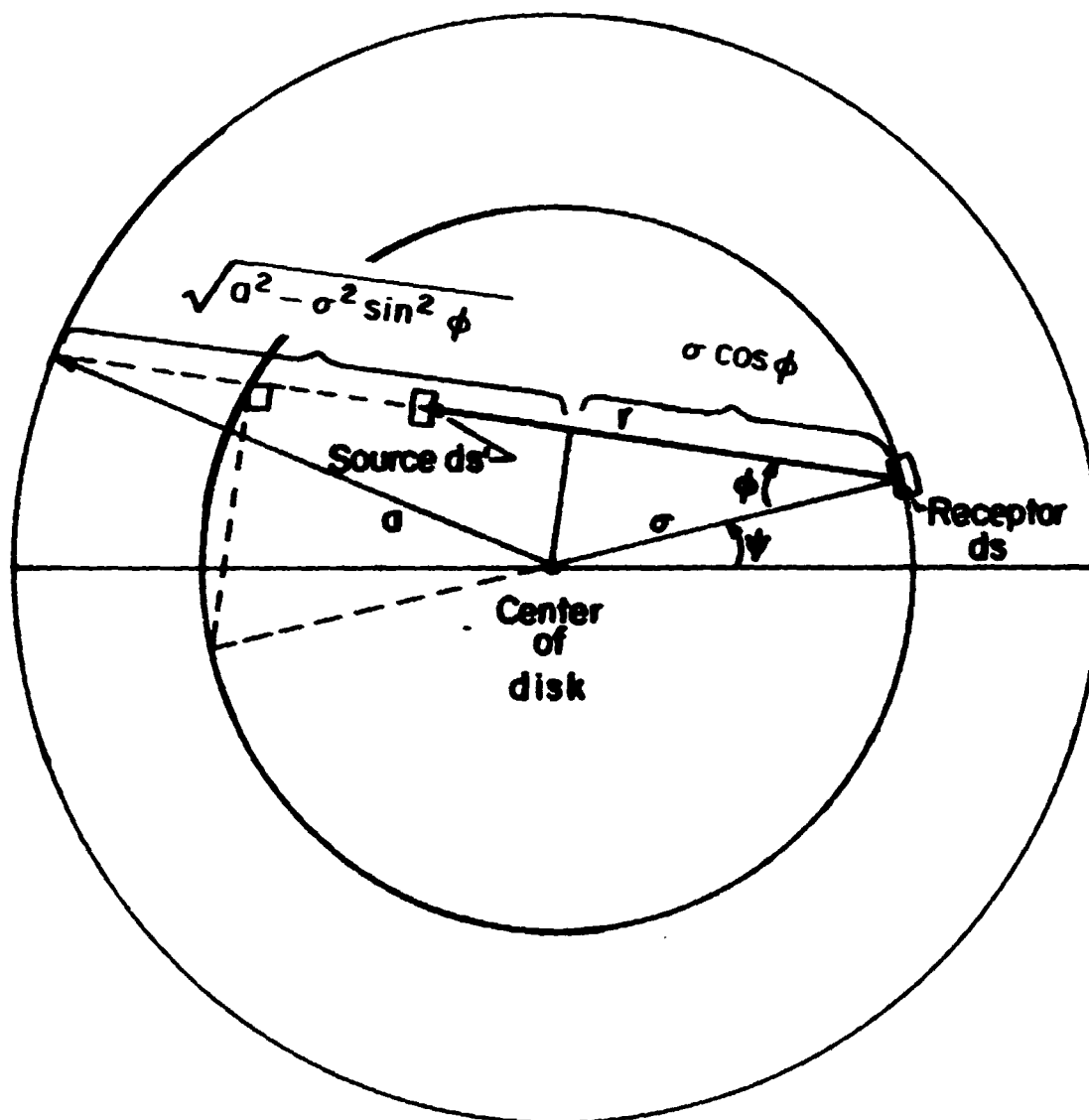


Figure 1.- Coordinates employed in describing the acoustic pressure at  $ds$  due to a source at  $ds'$  for a disk of radius  $a$ .



$$F = - \iint_{S'} \left[ \iint_S \frac{j \rho_0 c k \hat{u} e^{j(\omega t - kr)}}{4\pi r} ds \right] ds' \quad (3)$$

The coordinate system for performing the indicated integration over the front surface of the disk is shown in figure 1 (references 8, 9, and 10).

From figure 1, it can be seen that

$$r_{\max} = \sigma \cos \phi \pm [a^2 - \sigma^2 \sin^2 \phi]^{1/2} \quad (4)$$

It can also be seen by inspection of special cases, that the + sign is the correct selection for all quadrants given the convention for  $\phi$  as shown in figure 1. Hence for all situations

$$r_{\max} = \sigma \cos \phi + [a^2 - \sigma^2 \sin^2 \phi]^{1/2} \quad (5)$$

With this expression in the limit on  $r$  the pressure at  $ds$  due to all sources on the front surface of the disk becomes, in polar coordinates centered at  $ds$

$$p_i = \int_0^{2\pi} \int_0^{\sigma \cos \phi + [a^2 - \sigma^2 \sin^2 \phi]^{1/2}} \frac{j \rho_0 c k \hat{u} e^{j(\omega t - kr)}}{4\pi} dr d\phi \quad (6)$$

This equation can be written in a form where the limit is slightly modified as

$$P_i = \int_0^{2\pi} \int_0^a \left[ \frac{\sigma}{a} \cos \phi + \left( 1 - \left( \frac{\sigma}{a} \right)^2 \sin^2 \phi \right)^{1/2} \right] \frac{j \rho_0 c k \hat{u} e^{j(\omega t - kr)}}{4\pi} dr d\phi \quad (7)$$

The Green's function for one side of a disk radiating to the half space in an infinite baffle (reference 11) is

$$G_{\text{monopole}} = \frac{e^{j(\omega t - kr)}}{2\pi r} \quad (8)$$

#### Dipole Disk Theory

If the effect of the back face of the disk is also considered the pressure at the edge, i.e. when  $r = 2a \cos \phi$ , must be zero since the wave produced by one face is exactly  $180^\circ$  out of phase with the other and the geometries are identical. Thus we specify a general Green's function by

$$G_{\text{dipole}} = \frac{e^{j(\omega t - kr)}}{4\pi r} - \frac{e^{j(\omega t - k[4a \cos \phi - r])}}{4\pi[4a \cos \phi - r]} \quad (9)$$

It can be seen that this choice of a general Green's function does in fact produce the desired pressure at the edge.

There are several important points to be considered in relation to this Green's function.

1. The function  $G_{\text{dipole}}$  satisfies the characteristics of a Green's function; i.e., the first term on the right contains the

essential singularity while the first and second term together satisfy the boundary conditions and do not contain any singularities over the region.

2. With this choice of a Green's function the pressure can be determined by the application of Green's theorem since on the boundary both  $G_0$  and the outward normal gradient  $\frac{\partial G}{\partial n_0}$  are zero while the pressure is continuous. The line integrals over the boundary therefore vanish; and only the surface integral contributes to the pressure.

3. The physical significance of the Green's function is the following: The first term on the right of equation (9) is a monopole source located at any point on the front face of the disk. The second term is a monopole located along  $r$  on either the front or back face of the disk having strength just sufficient to cancel the source on the front face at the opposite boundary. Some examples are shown in figure 2.

The pressure on the front face of the disk is now given by integrating  $G$  over the surface of the front face

$$p = \frac{j \rho_0 c k \hat{u}}{4\pi} \int_0^{2\pi} \int_0^a \left[ \frac{a}{r} \cos \phi + \left( 1 - \left( \frac{a}{r} \right)^2 \sin^2 \phi \right)^{1/2} \right] \left[ \frac{e^{-jkr}}{r} - \frac{e^{-jk[4a \cos \phi - r]}}{[4a \cos \phi - r]} \right] r dr d\phi \quad (11)$$

By a change of variable let

$$x = \frac{r}{a} \quad (12)$$

also let

$$m = \frac{a}{a} \quad (13)$$

for convenience. These changes reduce the foregoing equation to the simpler form

$$p = \frac{j \rho_0 c \hat{u} e^{j\omega t}}{4\pi} (ka) \int_0^{2\pi} \int_0^{\pi} \left[ \frac{e^{-j(ka)x}}{x} - \frac{e^{-j(ka)[4 \cos \phi - x]}}{[4 \cos \phi - x]} \right] x \, dx \, d\phi \quad (14)$$

Since the second term in the integrand must be expanded in a Taylor's series before integration, there is no advantage to be gained in not expanding both terms. The expansions are, for  $(ka) \ll 1$ , i.e. very long wavelength

$$\frac{e^{-j(ka)x}}{x} = \left\{ \frac{1}{x} - \frac{(ka)^2 x}{2!} + \frac{(ka)^4 x^3}{4!} + \dots \right\} - j \left\{ \frac{(ka)}{1!} - \frac{(ka)^3 x^2}{3!} + \dots \right\} \quad (15)$$

and

$$\begin{aligned} \frac{e^{-j(ka)[4 \cos \phi - x]}}{[4 \cos \phi - x]} &= \left\{ \frac{1}{[4 \cos \phi - x]} - \frac{(ka)^2 [4 \cos \phi - x]}{2!} \right. \\ &\quad \left. + \frac{(ka)^4 [4 \cos \phi - x]^3}{4!} + \dots \right\} \\ &\quad - j \left\{ \frac{(ka)}{1!} - \frac{(ka)^3 [4 \cos \phi - x]^2}{3!} + \dots \right\} \quad (16) \end{aligned}$$

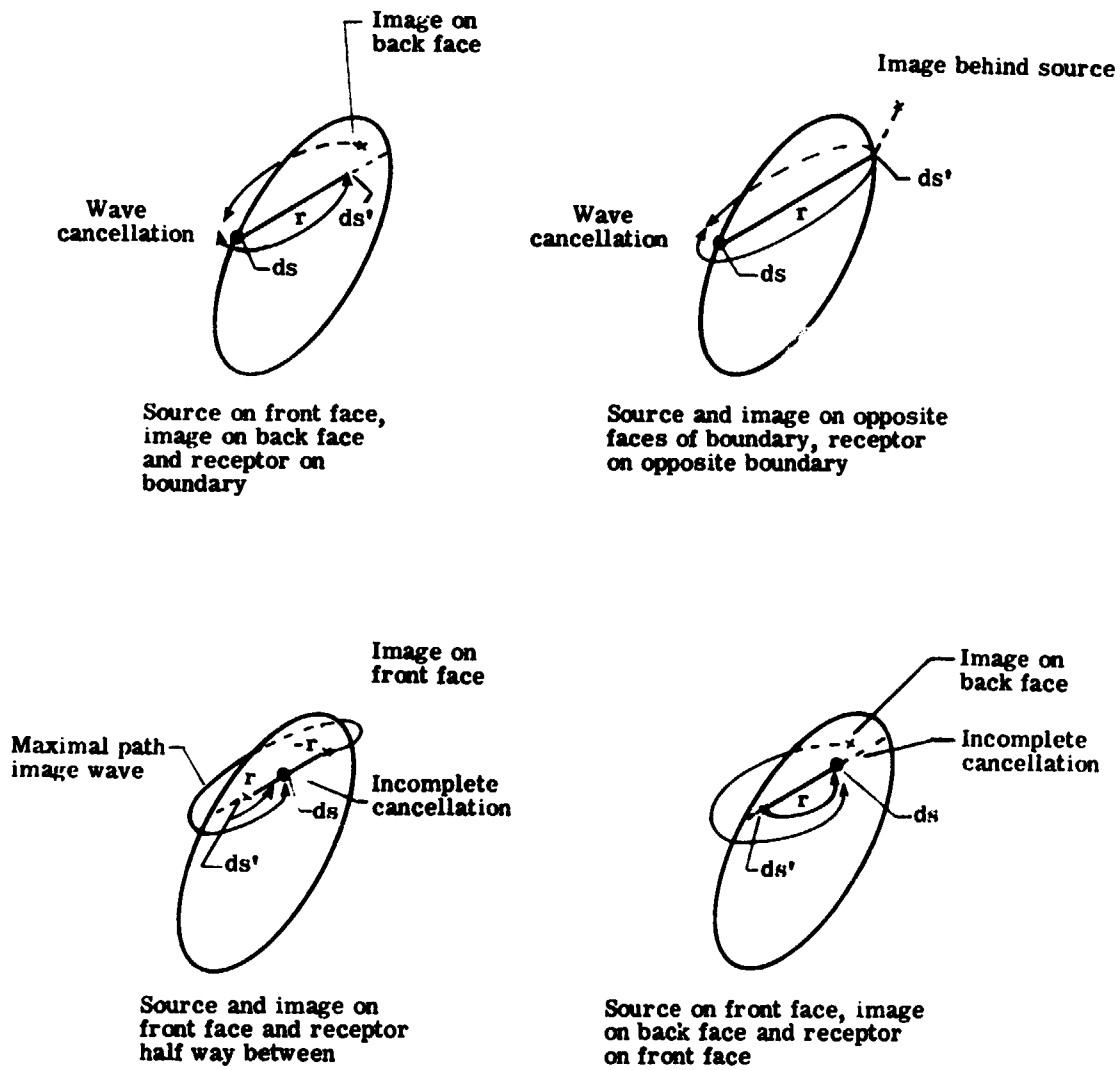


Figure 2.- Source and image arrangements for a dipole disk.

Hence, to sufficient accuracy

$$\begin{aligned}
 p = & \frac{j \rho_0 c \hat{u} e^{j\omega t}}{4\pi} (ka) \int_0^{2\pi} \int_0^{\frac{m \cos \phi + (1 - m^2 \sin^2 \phi)^{1/2}}{(4 \cos \phi - x)x} + \frac{(ka)^2}{2} (4 \cos \phi - 2x)} x \, dx \, d\phi \\
 & + \frac{j \rho_0 c \hat{u} e^{j\omega t}}{4\pi} (ka) \int_0^{2\pi} \int_0^{\frac{m \cos \phi + (1 - m^2 \sin^2 \phi)^{1/2}}{j \frac{(ka)^3}{3} [4x \cos \phi - 8 \cos^2 \phi]x}} dx \, d\phi
 \end{aligned} \tag{17}$$

The second integrand on the right above is real and corresponds to energy being radiated away from the disk. Both terms in the first integrand are imaginary and represent pressure due to attached mass. It can be seen that for  $ka \ll 1$  the second attached mass term is negligibly small in comparison to the first. However, as this term is larger in magnitude than the first radiation term it will be carried in the present analysis for the sake of completeness. The integrals will be taken one at a time.

#### First Attached Mass Term

The first term in equation (17) can be written

$$p_1 = \frac{j \rho_0 c \hat{u} e^{j\omega t}}{4\pi} (ka) \int_0^{2\pi} \int_0^{\frac{m \cos \phi + (1 - m^2 \sin^2 \phi)^{1/2}}{(4 \cos \phi - x)}} dx \, d\phi \tag{18}$$

The first integral of this can be found directly from tables and is given by

$$p_1 = \frac{j \rho_0 c \hat{u} e^{j\omega t}}{4\pi} (ka) \int_0^{2\pi} \left\{ 2m \cos \phi + 2(1 - m^2 \sin^2 \phi)^{1/2} \right. \\ \left. + 4 \cos \phi \ln \left| 1 - \frac{m \cos \phi + (1 - m^2 \sin^2 \phi)^{1/2}}{4 \cos \phi} \right| \right\} d\phi \quad (19)$$

Since an integral of the natural logarithm term in equation (19) is difficult it becomes expedient to expand this term to make the expression more tractable. An appropriate and highly accurate expansion is the standard form

$$\ln(1 - x) = - \left[ x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \dots \right] \quad (20)$$

$$[x^2 < 1 \text{ and } x \neq -1]$$

This expansion is accurate over most of the surface of the disk as can be seen from examination of figure 3. Also since  $x$  is a function of  $\cos \phi$  the integral of this expansion exists and is finite. Then

$$p_1 = \frac{j \rho c \hat{u} e^{j\omega t}}{4\pi} (ka) \int_0^{2\pi} \left\{ 2m \cos \phi + 2(1 - m^2 \sin^2 \phi)^{1/2} \right. \\ - m \cos \phi - (1 - m^2 \sin^2 \phi)^{1/2} - \frac{[m \cos \phi + (1 - m^2 \sin^2 \phi)^{1/2}]^2}{2(4 \cos \phi)} \\ \left. - \frac{[m \cos \phi + (1 - m^2 \sin^2 \phi)^{1/2}]^3}{3(4 \cos \phi)^2} + \text{higher order terms} \right\} d\phi \quad (21)$$

Raising terms to the indicated powers and carrying out the integration gives

$$p_1 = \frac{j \rho_0 c \hat{u} e^{j\omega t}}{4\pi} (ka) \left[ \left( \frac{49 - 12m - 5m^2}{12} \right) E\left(\frac{\pi}{2}, m\right) - \left( \frac{1 - m^2}{12} \right) K\left(\frac{\pi}{2}, m\right) \right] \quad (22)$$

Where  $K(\frac{\pi}{2}, m)$  is the complete elliptic integral of the first kind and where  $E(\frac{\pi}{2}, m)$  is the complete elliptic integral of the second kind. Now making use of the standard approximate expansions of the respective elliptic integrals

$$K\left(\frac{\pi}{2}, m\right) \approx \frac{\pi}{2} \left( 1 + \frac{m^2}{4} + \frac{9m^4}{64} + \dots \right) \quad (23)$$

and

$$E\left(\frac{\pi}{2}, m\right) \approx \frac{\pi}{2} \left( 1 - \frac{m^2}{4} - \frac{3m^4}{64} + \dots \right) \quad (24)$$

and carrying terms to the fourth power in  $m$

$$p_1 = \frac{j \rho_0 c \hat{u} e^{j\omega t}}{2} (ka) \left[ 1 - \frac{1}{4} m - \frac{11}{32} m^2 + \frac{1}{16} m^3 - \frac{5}{256} m^4 + \text{higher order terms} \right] \quad (25)$$

A plot of this pressure profile is given in figure 4 across a radius of the disk. In addition, a pressure profile for the back surface of the disk is identical to that of the front surface except for a  $180^\circ$



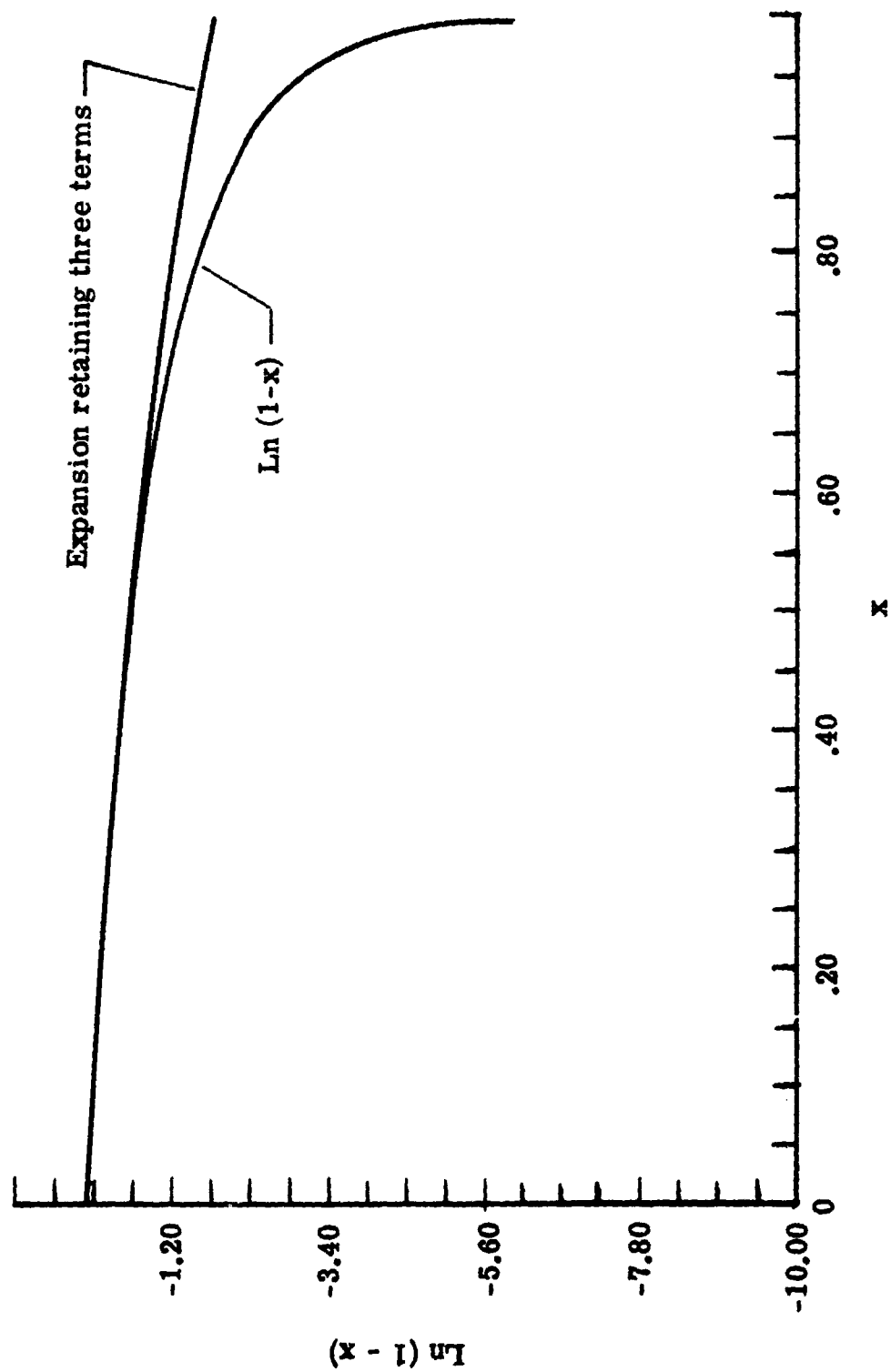


Figure 3.- Comparison of  $\ln(1-x)$  with  $-\left[x + \frac{x^2}{2} + \frac{x^3}{3}\right]$ .

phase shift. The total pressure difference is thus obtained by  
 do the pressure on one side.

### Second Attached Mass Term

The second term in equation (17) can be written

$$p_2 = \frac{j \rho_0 c \hat{u} e^{j\omega t}}{4\pi} (ka) \int_0^{2\pi} \int_0^{\pi} \frac{(ka)^2}{2} [4 \cos \phi - 2x] x \, dx \, d\phi. \quad (26)$$

The first integral of this is

$$p_2 = \frac{j \rho_0 c \hat{u} e^{j\omega t}}{4\pi} (ka)^3 \int_0^{2\pi} \left\{ \cos \phi [m \cos \phi + (1 - m^2 \sin^2 \phi)^{1/2}]^2 \right. \\ \left. - \frac{[m \cos \phi + (1 - m^2 \sin^2 \phi)^{1/2}]^3}{3} \right\} d\phi \quad (27)$$

Raising terms to the indicated powers and carrying out the indicated  
 integrations over  $\phi$  gives

$$p_2 = \frac{j \rho_0 c \hat{u} e^{j\omega t}}{\pi} (ka)^3 \left\{ \left[ \frac{2}{3m} - \frac{4}{9} \right] (1 + m^2) E\left(\frac{\pi}{2}, m\right) \right. \\ \left. - \frac{1}{3} (1 - m^2) E\left(\frac{\pi}{2}, m\right) - \left[ \frac{2}{3m} - \frac{4}{9} \right] (1 - m^2) K\left(\frac{\pi}{2}, m\right) \right\} \quad (28)$$

It can be seen that this term is smaller than the preceeding term  
 by a factor on the order of  $(ka)^2$  where  $ka \ll 1$  at low frequencies.  
 Hence this part of the pressure term is completely negligible in

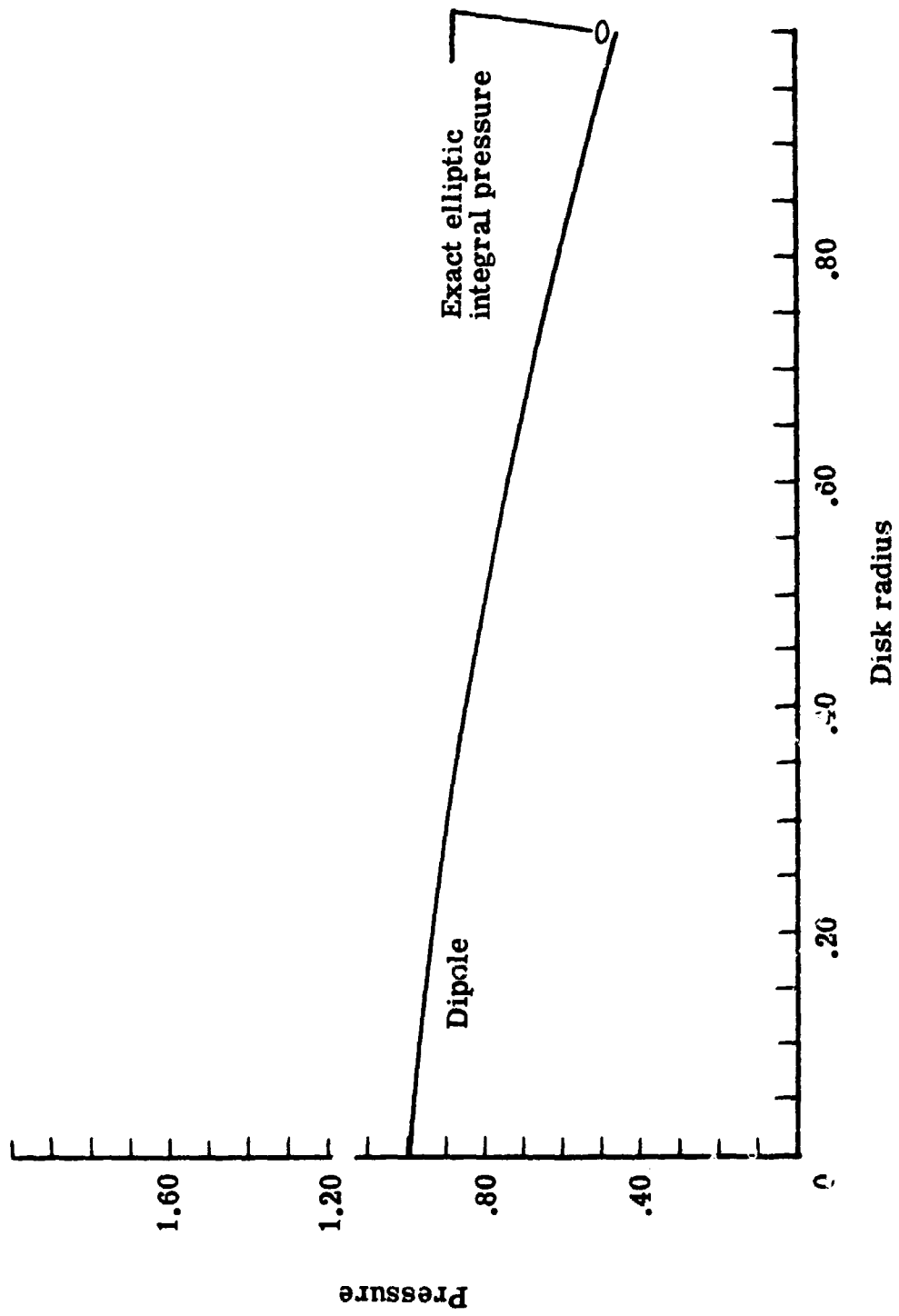


Figure 4.- Acoustic pressure of the "first attached mass term as a function of scaled disk radius.

in relation to the first attached mass term. If the elliptic integral expansions, equations (23) and (24), are employed as before and the results carried up to the fourth power in  $m$  the pressure becomes

$$p_2 = \frac{j \rho_0 c \hat{u} e^{j\omega t}}{2} (ka)^3 \left[ -\frac{1}{3} + m - \frac{1}{4} m^2 - \frac{1}{16} m^3 + \frac{1}{64} m^4 \right] \quad (29)$$

A scaled plot of equation (29) is shown in figure 5.

#### First Non-Cancelling Radiation Term

Strictly speaking the first radiation terms from the front and back faces of the disk cancel each other. In the present analysis this cancellation was done in the expansion of equation (14). Hence the normal monopole radiation, proportional to  $(ka)^2$ , is not present for a dipole. The first non-cancelling radiation term is therefore

$$p_3 = -\frac{\rho_0 c \hat{u} e^{j\omega t}}{3\pi} (ka)^4 \int_0^{2\pi} \int_0^1 [x \cos \phi - 2 \cos^2 \phi] x dx d\phi \quad (30)$$

The integral of this expression over  $x$  is

$$p_3 = -\frac{\rho_0 c \hat{u} e^{j\omega t}}{3\pi} (ka)^4 \int_0^{2\pi} \left\{ \frac{[m \cos \phi + (1 - m^2 \sin^2 \phi)^{1/2}]^3}{3} \cos \phi - [m \cos \phi + (1 - m^2 \sin^2 \phi)^{1/2}]^2 \cos^2 \phi \right\} d\phi \quad (31)$$

Squaring and cubing as indicated, and integrating again over  $2\pi$  gives

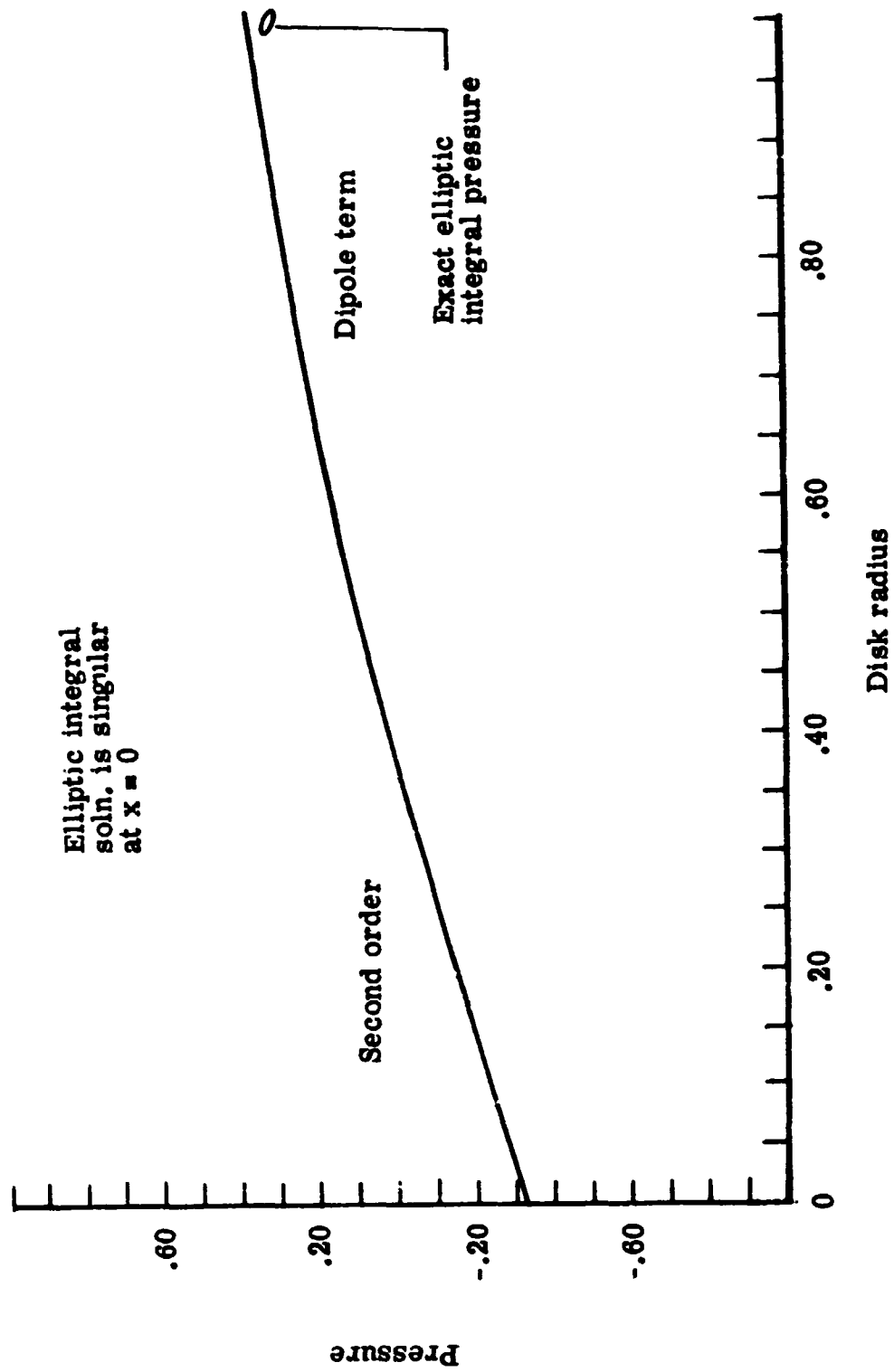


Figure 5.- Acoustic pressure due to the second attached mass term as a function of scaled disk radius.

$$p_3 = \frac{\rho_0 c \hat{u} e^{j\omega t}}{3} (ka)^4 \left[ 1 - m + \frac{1}{2} m^2 \right] \quad (32)$$

Elliptic integrals do not appear in the real terms.

It can be seen that for  $ka \ll 1$  this term is even smaller in magnitude than the previous two terms. However, as this is the first term which radiates energy away from the disk, it is important from a theoretical standpoint. A scaled plot of (32) is given in figure 6.

#### Total Pressure for a Disk in Translation

The total pressure for a disk in translation without a baffle is given by  $p_1 + p_2 + p_3$  and is explicitly, with  $\frac{\sigma}{a}$  for  $m$

$$p = \rho_0 c \hat{u} e^{j\omega t} \left\{ \frac{j(ka)}{4\pi} \left[ \left( \frac{49 - 12 \left( \frac{\sigma}{a} \right) - 5 \left( \frac{\sigma}{a} \right)^2}{12} \right) E\left(\frac{\pi}{2}, \frac{\sigma}{a}\right) - \left( \frac{1 - \left( \frac{\sigma}{a} \right)^2}{12} \right) K\left(\frac{\pi}{2}, \frac{\sigma}{a}\right) \right] + \frac{j(ka)^3}{\pi} \left[ \left( \frac{2a}{3\sigma} - \frac{4}{9} \right) \left( 1 + \left( \frac{\sigma}{a} \right)^2 \right) E\left(\frac{\pi}{2}, \frac{\sigma}{a}\right) - \frac{1}{3} \left( 1 - \left( \frac{\sigma}{a} \right)^2 \right) E\left(\frac{\pi}{2}, \frac{\sigma}{a}\right) - \left( \frac{2a}{3\sigma} - \frac{4}{9} \right) \left( 1 - \left( \frac{\sigma}{a} \right)^2 \right) K\left(\frac{\pi}{2}, m\right) \right] + \frac{(ka)^4}{3} \left[ 1 - \left( \frac{\sigma}{a} \right) + \frac{1}{2} \left( \frac{\sigma}{a} \right)^2 \right] \right\} \quad ka \ll 1 \quad (33)$$

This same equation with the elliptic integrals expanded and carried to the fourth power in  $\left( \frac{\sigma}{a} \right)$  is

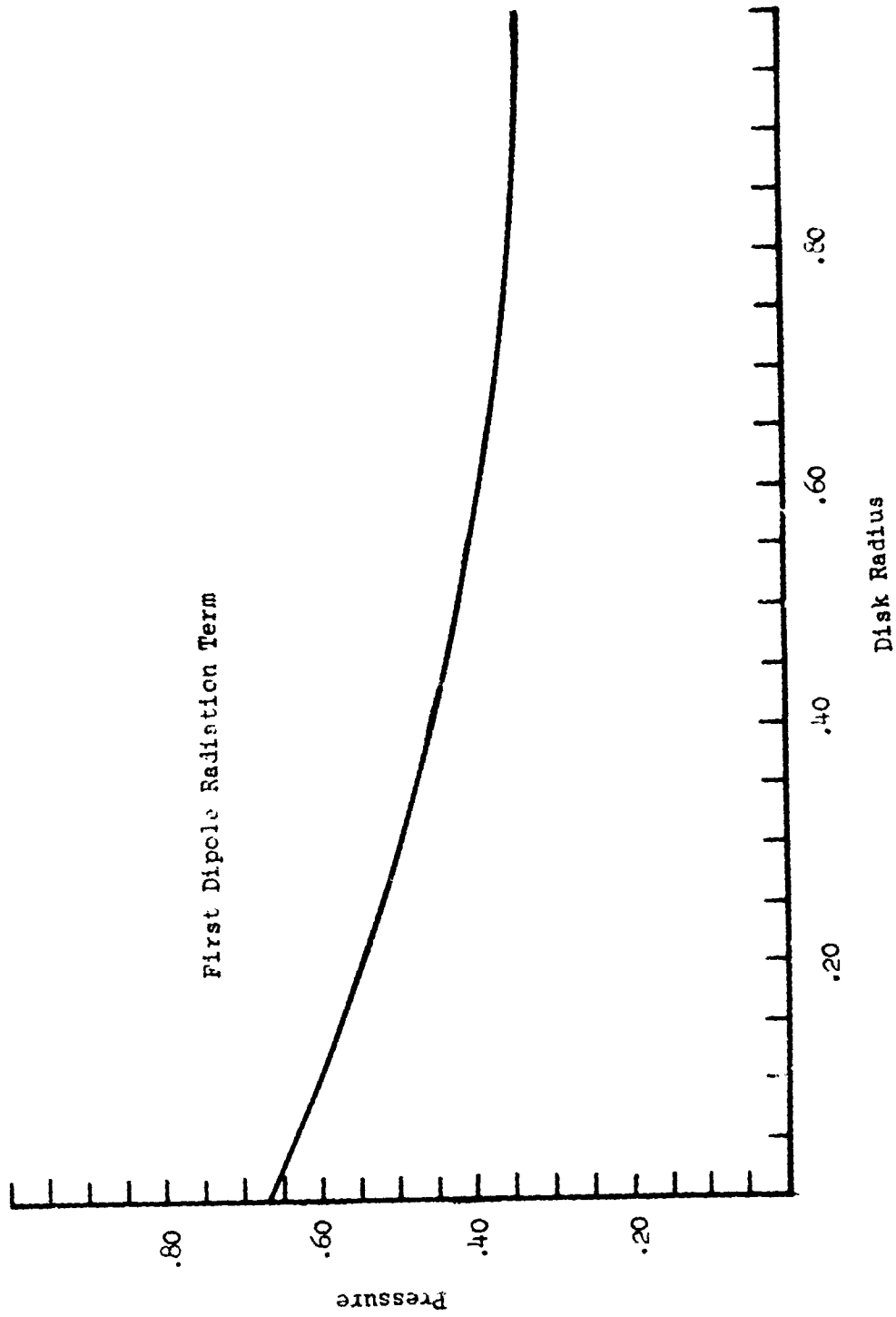


Figure 6.- Radiation pressure due to the first non-vanishing radiation term as a function of scaled disk radius.

$$\begin{aligned}
p = \rho_0 \, c \hat{u} \, e^{j\omega t} & \left\{ \frac{j(ka)}{2} \left[ 1 - \frac{1}{4} \left( \frac{\sigma}{a} \right) - \frac{11}{32} \left( \frac{\sigma}{a} \right)^2 + \frac{1}{16} \left( \frac{\sigma}{a} \right)^3 + \frac{5}{256} \left( \frac{\sigma}{a} \right)^4 \right] \right. \\
& + j \frac{(ka)^3}{2} \left[ -\frac{1}{3} + \left( \frac{\sigma}{a} \right) - \frac{1}{4} \left( \frac{\sigma}{a} \right)^2 - \frac{1}{16} \left( \frac{\sigma}{a} \right)^3 + \frac{1}{64} \left( \frac{\sigma}{a} \right)^4 \right] \\
& \left. + \frac{(ka)^4}{3} \left[ 1 - \left( \frac{\sigma}{a} \right) + \frac{1}{2} \left( \frac{\sigma}{a} \right)^2 \right] \right\} \quad (34)
\end{aligned}$$

#### Force on a Disk in Translation.

The force on one face of the disk is obtained by integrating the pressure over the surface of the disk. Using equation (34)

$$\begin{aligned}
F = \rho_0 \, c \hat{u} \, e^{j\omega t} & \int_0^{2\pi} \int_0^a \left\{ \frac{j(ka)}{2} \left[ 1 - \frac{1}{4} \left( \frac{\sigma}{a} \right) - \frac{11}{32} \left( \frac{\sigma}{a} \right)^2 + \frac{1}{16} \left( \frac{\sigma}{a} \right)^3 - \frac{5}{256} \left( \frac{\sigma}{a} \right)^4 \right] \right. \\
& + j \frac{(ka)^3}{2} \left[ -\frac{1}{3} + \left( \frac{\sigma}{a} \right) - \frac{1}{4} \left( \frac{\sigma}{a} \right)^2 - \frac{1}{16} \left( \frac{\sigma}{a} \right)^3 + \frac{1}{64} \left( \frac{\sigma}{a} \right)^4 \right] \\
& \left. + \frac{(ka)^4}{3} \left[ 1 - \left( \frac{\sigma}{a} \right) + \frac{1}{2} \left( \frac{\sigma}{a} \right)^2 \right] \right\} \sigma \, d\sigma \, d\psi \quad (35)
\end{aligned}$$

With the indicated operations carried out, the force is

$$F = \rho_0 \, c \hat{u} \, e^{j\omega t} \, a^2 \{ j[1.0681 (ka) + 0.2962 (ka)^3] + [0.6109 (ka)^4] \} \quad (36)$$

Since the two faces of the disk are out of phase the total force is twice this, or

$$F = \rho_0 \, c \hat{u} \, e^{j\omega t} \, a^2 \{ j[2.1362 (ka) + 0.5924 (ka)^3] + [1.2218 (ka)^4] \} \quad (37)$$



## Acoustic Impedance

The radiation impedance is given by

$$Z_r = \frac{F}{\hat{u} e^{j\omega t}} \quad (38)$$

Hence the radiation impedance of the disk is given by

$$Z_r = \rho_0 c a^2 \{j[2.1362 (ka) + 0.5924 (ka)^3] + [1.2218 (ka)^4]\} \quad (39)$$

as a point of comparison, the radiation impedance of a simple dipole is well known. In terms of the present notation this impedance is given (ref. 11, p. 317) by

$$Z_r \approx \rho_0 c a^2 \{j[2.0943 (ka) + 1.0472 (ka)^3] + [1.047 (ka)^4]\} \quad (40)$$

It can be seen that there is reasonable agreement between the disk, which is a distributed dipole, and a simple dipole at low frequency as would be expected, with the disk slightly the higher of the two.

## Apparent Mass

The apparent mass is given by the non-radiating, or imaginary part of the reaction to the force.

$$F = - \frac{d}{dt} (m_a v) \quad (41)$$

Or, if the apparent mass is assumed constant

$$F = - m_a \frac{dv}{dt} \quad (42)$$

In the case of a disk in translation

$$\frac{dv}{dt} = - j \omega \hat{u} e^{j\omega t} \quad (43)$$

Therefore, using equations (42), (43), and the first imaginary term of (37)

$$m_a = 2.1362 \rho_0 a^3 \quad (44)$$

#### Pressure on a Disk Pendulum

That the effective mass and inertia of a disk pendulum are greater in air than in a vacuum can be readily demonstrated experimentally. However, in the past, these effects have generally been measured mechanically (ref. 12 and 13) and theoretical treatment has been by means of classical hydrodynamics. The result led to some difficulty in visualizing the relative part played by rotational terms and translational terms, especially when the pendulum is of intermediate length so that rotation and translation contribute significantly to the total motion. An alternative approach is taken in the present paper; viz., to consider the pendulum as an acoustic source, albeit of extremely low frequency.

Consider a pendulum which consists of a disk suspended by a massless support and swinging in still air about an axis in the plane of the disk, so that the air impinges on the flat face of the disk. If the motion of the pendulum is undamped, as it would be if the pendulum were suspended in a vacuum, then the equation of motion for small amplitude oscillation is

$$I\ddot{\theta} + m_s g l_{c.g.} \theta = 0 \quad (45)$$

The period of oscillation is given by the well known formula

$$P = 2\pi \sqrt{\frac{I}{m_s g l_{c.g.}}} \quad (46)$$

with circular angular velocity

$$\omega = \sqrt{\frac{m_s g l_{c.g.}}{I}} \quad (47)$$

In contrast, the damping moment acting on a pendulum oscillating in still air is proportional to the drag (fig. 7) which, in turn, is proportional to the square of the velocity. In this case, the drag force is given by the product of the aerodynamic pressure  $\frac{1}{2} \rho V^2$ , the surface area of the pendulum on which this pressure acts  $S$ , and the drag coefficient of the pendulum  $C_D$ . The damping moment is the product of the drag force and the moment arm. Hence, for small amplitudes, the equation of motion with drag added is given by

$$I\ddot{\theta} + \frac{1}{2} \rho V |V| S C_D l_{\text{aero}} + m_s g l_{\text{c.g.}} \theta = 0 \quad (48)$$

The moment of inertia  $I$  includes the structural and apparent moments of inertia. However, the restoring moment is related to the structural mass  $m_s$  only. The apparent mass produces no restoring force, since the buoyant force balances the gravity force.

The velocity term in equation (48) is given by

$$V = l_{\text{aero}} \dot{\theta} \quad (49)$$

Therefore, equation (48) can be written as

$$I\ddot{\theta} + \frac{1}{2} \rho (l_{\text{aero}})^3 S C_D \dot{\theta} |\dot{\theta}| + m_s g l_{\text{c.g.}} \theta = 0 \quad (50)$$

It should be noted that this equation differs from the more commonly encountered equation of damped oscillatory motion in which the damping is proportional to the first power of the velocity.

Equation (50) is a nonlinear, second-order, differential equation which does not have a known solution. However, the behavior of this equation has been examined numerically for a wide range of constant damping coefficients. A typical solution is shown in figure 8 which was taken from reference 13. It can be seen that the period is nearly constant, is independent of the damping, and is equal to the period which would be predicted on the assumption that no

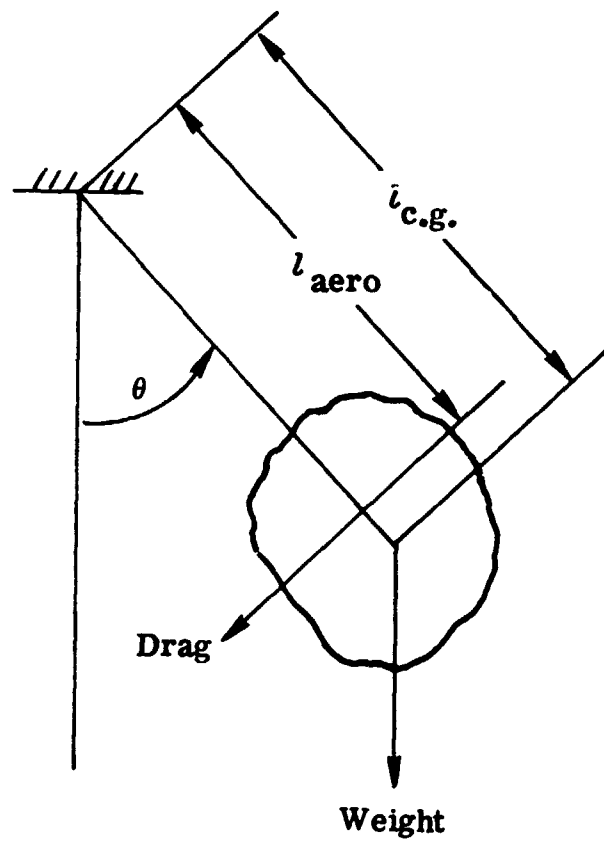


Figure 7.- Physical pendulum showing relative orientation of principal forces and distances.

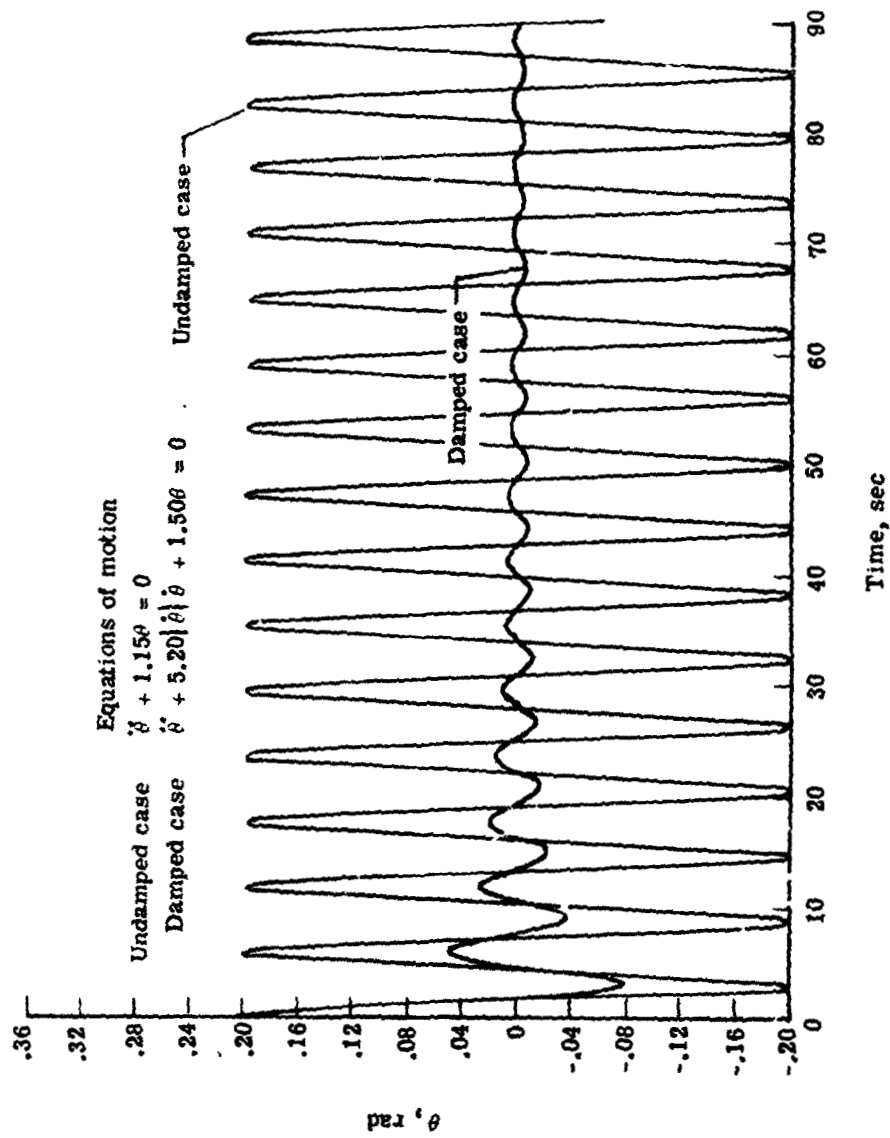


Figure 8.- Typical time histories (eq. (4)) for damped and undamped oscillations. Initial conditions:  $\theta_0 = 0.2$  rad,  $\dot{\theta}_0 = 0$  rad/sec.

damping is present at all. The same result was obtained in the approximate solution of equation (50) given in appendix B. These results have also been verified experimentally by the author (not published). The results, therefore, indicate that, if the damping is proportional to the square of the velocity, the period is essentially the same as that for the undamped situation. Hence, the circular velocity is given to sufficient accuracy by equation (47). The value of this is that the period in air gives a measure of the attached mass. In this equation  $m_s$ ,  $g$ , and  $l_{c.g.}$  are constants. For purposes of this analysis, it will be assumed that  $I$  depends only on the mass and geometric parameters of the pendulum and on the density of the attached air. Hence  $I$  can also be considered constant and  $\omega$  is constant. The disk of the pendulum may, therefore, be considered as a distributed acoustic source oscillating at angular rate  $\omega$ . Two possible types of motion will be present, viz. translation and rotation. Both types of motion will produce pressure effects on the face of the disk. These pressure effects can be broken up into forces and torques in the following manner:

#### Pressure Across the Disk of the Pendulum

The maximum speed  $\hat{u}$  of any infinitesimal point on the surface of the disk is determined by its distance from the pendulum support point. Thus if  $\hat{\theta}$  is the maximum angular speed of the pendulum, i.e.; as it passes through bottom dead center

$$\hat{u} = (\ell_{c.g.} - b) \dot{\hat{\theta}} \quad (51)$$

where  $b$  is the distance of the point where the pressure is applied from a horizontal line through the center of gravity of the disk (figure 9).

From figure 9

$$b = r \sin(\phi - \psi) + \sigma \sin \psi$$

Then (51) becomes

$$\hat{u} = \ell_{c.g.} \dot{\hat{\theta}} - \sigma \sin \psi \dot{\hat{\theta}} - r \sin(\phi - \psi) \dot{\hat{\theta}} \quad (51a)$$

This equation can be inserted in equation (11) so that

$$p_i = \int_0^{2\pi} \int_0^a \frac{j \rho_0 c k [\ell_{c.g.} \dot{\hat{\theta}} - \sigma \sin \psi \dot{\hat{\theta}} - r \sin(\phi - \psi) \dot{\hat{\theta}}]}{4\pi} e^{j\omega t} \left[ \frac{e^{-jkr}}{r} - \frac{e^{-jk[4a \cos \phi - r]}}{[4a \cos \phi - r]} \right] r dr d\phi \quad (52)$$

This can be broken into two integrals,



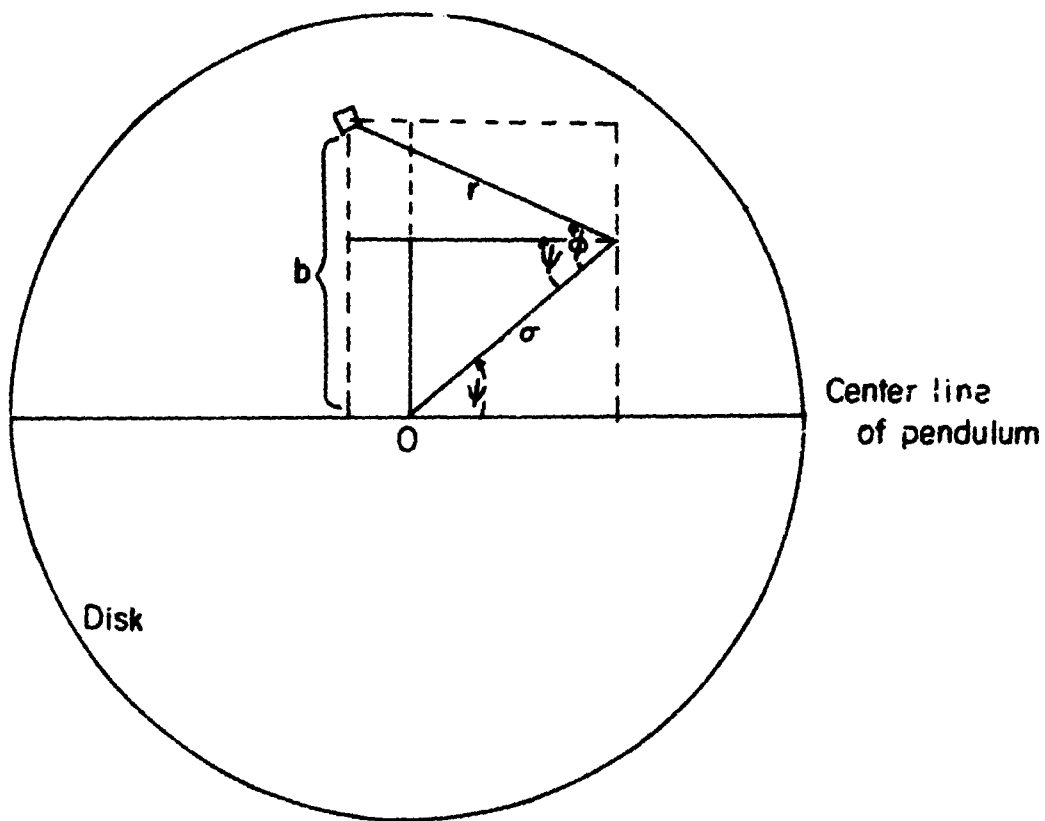


Figure 9.- Coordinates used to specify speed as a function of distance from the center line of pendulum.

$$\begin{aligned}
p_i = & \int_0^a \int_0^{2\pi} \frac{j \rho_0 c k [\ell_{c.g.} \hat{\theta} - \sigma \sin \psi \hat{\theta}] e^{j\omega t}}{4\pi} \left[ \frac{e^{-jkr}}{r} - \frac{e^{-jk[4a \cos \phi - r]}}{[4a \cos \phi - r]} \right] r dr d\phi \\
& - \int_0^a \int_0^{2\pi} \frac{j \rho_0 c k [r \sin(\phi - \psi) \hat{\theta}] e^{j\omega t}}{4\pi} \left[ \frac{e^{-jkr}}{r} - \frac{e^{-jk[4a \cos \phi - r]}}{[4a \cos \phi - r]} \right] r dr d\phi
\end{aligned} \quad (53)$$

The first of these two integrals is seen to be the same as equation (11) except for the substitution of the term  $[\ell_{c.g.} \hat{\theta} - \sigma \sin \psi \hat{\theta}]$  in place of the constant  $\hat{u}$ . Hence the integral of this expression is merely equation (33) or its expanded form (34) with  $[\ell_{c.g.} \hat{\theta} - \sigma \sin \psi \hat{\theta}]$  replacing  $\hat{u}$ . That is, part of the pendulum pressure profile has the same form as that of the simple disk developed previously. If (53) is written in the convenient form

$$p_i = p_I + p_{II} \quad (54)$$

with  $p_I$  analogous to (33)

$$\begin{aligned}
p_I = & \rho_0 c [\ell_{c.g.} \hat{\theta} - \sigma \sin \psi \hat{\theta}] e^{j\omega t} \left\{ \frac{j(ka)}{2} \left[ 1 - \frac{1}{4} \left( \frac{\sigma}{a} \right) - \frac{11}{32} \left( \frac{\sigma}{a} \right)^2 \right. \right. \\
& + \left. \frac{1}{16} \left( \frac{\sigma}{a} \right)^3 + \frac{5}{256} \left( \frac{\sigma}{a} \right)^4 \right] + \frac{j(ka)^3}{2} \left[ -\frac{1}{3} + \left( \frac{\sigma}{a} \right) - \frac{1}{4} \left( \frac{\sigma}{a} \right)^3 - \frac{1}{16} \left( \frac{\sigma}{a} \right)^3 + \frac{1}{64} \left( \frac{\sigma}{a} \right)^4 \right] \\
& \left. + \frac{(ka)^4}{3} \left[ 1 - \left( \frac{\sigma}{a} \right) + \frac{1}{2} \left( \frac{\sigma}{a} \right)^2 \right] \right\}
\end{aligned} \quad (55)$$

and

$$P_{II} = \int_0^a \int_0^{2\pi} \frac{j \rho_0 c \hat{\theta} e^{j\omega t}}{4\pi} \left[ \frac{e^{-jkr}}{r} - \frac{e^{-jk[4a \cos \phi - r]}}{[4a \cos \phi - r]} \right] r^2 \sin(\phi - \psi) dr d\phi \quad (56)$$

Again there is no particular advantage to be gained in integrating the first term in the square bracket directly so both terms are expanded in a Taylor's series. First change the variables by letting

$$x = \frac{r}{a} \quad (12)$$

and

$$m = \frac{\sigma}{a} \quad (13)$$

as was done previously. Then equation (56) becomes

$$P_{II} = - \frac{j \rho_0 c \hat{\theta} e^{j\omega t}}{4\pi} ka^2 \int_0^a \int_0^{2\pi} \sin(\phi - \psi) \left[ \frac{e^{-j(ka)x}}{x} - \frac{e^{-j(ka)[4 \cos \phi - x]}}{[4 \cos \phi - x]} \right] x^2 dx d\phi \quad (57)$$

The term in brackets can now be expanded as before with the result that

$$\begin{aligned}
P_{II} = & - \frac{j \rho_0 c \hat{\theta} e^{j\omega t}}{4\pi} ka^2 \int_0^{2\pi m \cos \phi + (1 - m^2 \sin^2 \phi)^{1/2}} \int_0^{\sin(\phi-\psi) \left[ \frac{(4 \cos \phi - 2x)}{x(4 \cos \phi - x)} \right.} \\
& \left. + \frac{(ka)^2}{2} (4 \cos \phi - 2x) \right] x^2 dx d\phi \\
& - \frac{j \rho_0 c \hat{\theta} e^{j\omega t}}{4\pi} ka^2 \int_0^{2\pi m \cos \phi + (1 - m^2 \sin^2 \phi)^{1/2}} \int_0^{\sin(\phi-\psi) \frac{1(ka)^3}{3} [4 x \cos \phi - 8 \cos^2 \phi]} x^2 dx d\phi
\end{aligned} \tag{58}$$

Again there are two imaginary terms in lower powers of  $(ka)$  before reaching the first real radiation term. Again, the integrals will be taken one at a time. Designating these integrals as  $P_4$ ,  $P_5$  and  $P_6$  respectively, one has the following.

#### First Attached Mass Rotation Term

The first term is

$$P_4 = - \frac{j \rho_0 c \hat{\theta} e^{j\omega t}}{4\pi} ka^2 \int_0^{2\pi m \cos \phi + (1 - m^2 \sin^2 \phi)^{1/2}} \int_0^{\sin(\phi-\psi) \left[ \frac{(4 \cos \phi - 2x)}{x(4 \cos \phi - x)} \right]} x^2 dx d\phi \tag{59}$$

The first integral of this gives

$$p_4 = - \frac{j \rho_0 c \hat{\theta} e^{j\omega t}}{4\pi} k a^2 \int_0^{2\pi} \sin(\phi - \psi) \left[ x_m^2 + 4x_m \cos \phi + (4 \cos \phi)^2 \ln \left| 1 - \frac{x_m}{4 \cos \phi} \right| \right] d\phi \quad (60)$$

where  $x_m$  is the upper limit

$$x_m = m \cos \phi + (1 - m^2 \sin^2 \phi)^{1/2} \quad (61)$$

as was done in equation (20)  $\ln(1 - x)$  is expanded

$$\ln(1 - x) = - \left[ x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \dots \right] \quad (62)$$

$$[x^2 < 1 \text{ and } x = -1]$$

This expansion is accurate over most of the surface of the disk as can be seen from examination of figure 3. Hence to three term accuracy

$$p_4 = - \frac{j \rho_0 c \hat{\theta} e^{j\omega t}}{4\pi} k a^2 \int_0^{2\pi} \sin(\phi - \psi) \left[ x_m^2 + 4x_m \cos \phi - (4 \cos \phi)^2 \left\{ \frac{x_m}{(4 \cos \phi)} + \frac{x_m^2}{2(4 \cos \phi)^2} + \frac{x_m^3}{3(4 \cos \phi)^3} + \text{higher order terms} \right\} \right] d\phi \quad (63)$$

A cancellation in the  $x_m$  term yields

$$p_4 = - \frac{j \rho_0 c \hat{\theta} e^{j\omega t}}{4\pi} k a^2 \int_0^{2\pi} \sin(\phi - \psi) \left[ \frac{x_m^2}{2} - \frac{x_m^3}{3(4 \cos \phi)} \right] d\phi \quad (64)$$

If  $x_m$  is written explicitly in terms of  $\phi$  and use is made of the identity

$$\sin(\phi - \psi) \equiv \sin \phi \cos \psi - \cos \phi \sin \psi \quad (65)$$

the indicated integration over  $2\pi$  can be performed. The result is

$$p_4 = \frac{j \rho_0 c \hat{\theta} e^{j\omega t}}{4\pi} k a^2 \left\{ \left( \frac{12 - 7m + 12m^2 - m^3}{9m} \right) E\left(\frac{\pi}{2}, m\right) - \left( \frac{12 - 4m - 12m^2 + 4m^3}{9m} \right) K\left(\frac{\pi}{2}, m\right) \right\} \sin \psi \quad (66)$$

If the elliptic integrals are expanded by (23) and (24) and terms up to the fourth power in  $m$  are retained the above equation becomes

$$p_4 = \frac{j \rho_0 c \hat{\theta} e^{j\omega t}}{8} k a^2 \left\{ -\frac{1}{3} + 2m - \frac{1}{4}m^2 - \frac{1}{4}m^3 + \frac{1}{64}m^4 \right\} \sin \psi \quad (67)$$

#### Second Attached Mass Rotation Term

The second term is

$$p_5 = - \frac{j \rho_0 c \hat{\theta} e^{j\omega t}}{4\pi} k a^2 \int_0^{2\pi} \int_0^{\frac{m \cos \phi + (1 - m^2 \sin^2 \phi)^{1/2}}{2}} \sin(\phi - \psi) \left[ \frac{(ka)^2}{2} (4 \cos \phi - 2x) \right] x^2 dx d\phi \quad (68)$$

The first integral of this expression is

$$p_5 = - \frac{j \rho_0 c \hat{\theta} e^{j\omega t}}{8\pi} (ka)^3 a \int_0^{2\pi} \sin(\phi - \psi) \left[ \frac{4}{3} \cos \phi x_m^3 - \frac{1}{2} x_m^4 \right] d\phi \quad (69)$$

where

$$x_m = m \cos \phi + (1 - m^2 \sin^2 \phi)^{1/2}$$

as before.

Then writing (69) explicitly in terms of  $\phi$  and making use of (65), (69) can be integrated. The result is

$$p_5 = - \frac{j \rho_0 c \hat{\theta} e^{j\omega t}}{2\pi} \frac{(ka)^3 a}{45 m^2} [(12 + 6m - 112m^2 + 84m^3 - 28m^4 + 6m^5) E(\frac{\pi}{2}, m) - (12 + 6m - 88m^2 + 36m^3 + 76m^4 - 42m^5) K(\frac{\pi}{2}, m)] \sin \psi \quad (70)$$

as before, the elliptic integrals can be expanded and terms up to  $m^4$  retained to yield

$$p_5 = j \rho_0 c \hat{\theta} e^{j\omega t} (ka)^3 a \left[ \frac{1}{6} - \frac{1}{4} m + \frac{5}{8} m^2 - \frac{9}{76} m^3 + \frac{1}{180} m^4 \right] \sin \psi \quad (71)$$

## First Radiation Term in Rotation (That Does Not Cancel)

The first non-cancelling radiation term is

$$p_6 = - \frac{\rho_0 \hat{c} \hat{\theta} e^{j\omega t}}{3\pi} (ka)^4 a \int_0^{2\pi} \int_0^{\frac{m \cos \phi + (1 - m^2 \sin^2 \phi)^{1/2}}{m}} \sin(\phi - \psi) [x^3 \cos \phi - 2x^2 \cos^2 \phi] dx d\phi \quad (72)$$

The first integral of this is

$$p_6 = - \frac{\rho_0 \hat{c} \hat{\theta} e^{j\omega t}}{3\pi} (ka)^4 a \int_0^{2\pi} \sin(\phi - \psi) \left[ \frac{x_m^4}{4} \cos \phi - \frac{2}{3} x_m^3 \cos^2 \phi \right] d\phi \quad (73)$$

with

$$x_m = m \cos \phi + (1 - m^2 \sin^2 \phi)^{1/2}$$

As before, making use of (65) and writing (73) explicitly in terms of  $\phi$ , the equation can be integrated. The result is

$$p_6 = \frac{\rho_0 \hat{c} \hat{\theta} e^{j\omega t}}{3} (ka)^4 a \left[ \frac{1}{4} - \frac{3}{2} m + m^2 + \frac{1}{3} m^3 \right] \sin \psi \quad (74)$$

as usual, the radiation term does not contain elliptic integrals.

## Pressure Distribution of a Pendulum

The final result of the foregoing analysis is the total pressure distribution at low frequency of a disk pendulum of arbitrary size and length. One has merely to add up the pressures being careful to substitute  $[\ell_{c.g.} - \sigma \sin \psi] \hat{\theta}$  in place of  $\hat{u}$  in  $p_1$ ,  $p_2$ , and  $p_3$ .



The pressure is

$$\begin{aligned}
 p = & \frac{j \rho_0 c [\ell_{c.g.} - \sigma \sin \psi] \hat{\theta} e^{j\omega t}}{4\pi} (ka) \left[ \left( \frac{49 - 12\left(\frac{\sigma}{a}\right) - 5\left(\frac{\sigma}{a}\right)^2}{12} \right) E\left(\frac{\pi}{2}, \frac{\sigma}{a}\right) \right. \\
 & \left. - \frac{(1 - \left(\frac{\sigma}{a}\right)^2)}{12} K\left(\frac{\pi}{2}, \frac{\sigma}{a}\right) \right] + \frac{j \rho_0 c \hat{\theta} e^{j\omega t}}{4\pi} (ka) a \left[ \left( \frac{12 - 7\left(\frac{\sigma}{a}\right) + 12\left(\frac{\sigma}{a}\right)^2 - \left(\frac{\sigma}{a}\right)^3}{9\left(\frac{\sigma}{a}\right)} \right) E\left(\frac{\pi}{2}, \frac{\sigma}{a}\right) \right. \\
 & \left. - \left( \frac{12 - 4\left(\frac{\sigma}{a}\right) - 12\left(\frac{\sigma}{a}\right)^2 + 4\left(\frac{\sigma}{a}\right)^3}{9\left(\frac{\sigma}{a}\right)} \right) K\left(\frac{\pi}{2}, \frac{\sigma}{a}\right) \right] \sin \psi \\
 & + \frac{j \rho_0 c [\ell_{c.g.} - \sigma \sin \psi] \hat{\theta} e^{j\omega t}}{\pi} (ka)^3 \left[ \left( \frac{2}{3\left(\frac{\sigma}{a}\right)} - \frac{4}{9} \right) \left( 1 + \left(\frac{\sigma}{a}\right)^2 \right) E\left(\frac{\pi}{2}, \frac{\sigma}{a}\right) \right. \\
 & \left. - \frac{1}{3} \left( 1 - \left(\frac{\sigma}{a}\right)^2 \right) E\left(\frac{\pi}{2}, \frac{\sigma}{a}\right) - \left[ \frac{2}{3\left(\frac{\sigma}{a}\right)} - \frac{4}{9} \right] \left( 1 - \left(\frac{\sigma}{a}\right)^2 \right) K\left(\frac{\pi}{2}, \frac{\sigma}{a}\right) \right] \\
 & - \frac{j \rho_0 c \hat{\theta} e^{j\omega t}}{2\pi} \frac{(ka)^3 a}{45\left(\frac{\sigma}{a}\right)^2} \left[ \left( 12 + 6\left(\frac{\sigma}{a}\right) - 112\left(\frac{\sigma}{a}\right)^2 + 84\left(\frac{\sigma}{a}\right)^3 - 28\left(\frac{\sigma}{a}\right)^4 \right. \right. \\
 & \left. \left. + 6\left(\frac{\sigma}{a}\right)^5 \right) E\left(\frac{\pi}{2}, \frac{\sigma}{a}\right) - \left( 12 + 6\left(\frac{\sigma}{a}\right) - 88\left(\frac{\sigma}{a}\right)^2 + 36\left(\frac{\sigma}{a}\right)^3 + 76\left(\frac{\sigma}{a}\right)^4 \right. \right. \\
 & \left. \left. - 42\left(\frac{\sigma}{a}\right)^5 \right) K\left(\frac{\pi}{2}, \frac{\sigma}{a}\right) \right] \sin \psi + \frac{\rho_0 c [\ell_{c.g.} - \sigma \sin \psi]}{3} e^{j\omega t} (ka)^4 \left[ 1 - \left(\frac{\sigma}{a}\right) + \frac{1}{2}\left(\frac{\sigma}{a}\right)^2 \right] \\
 & + \frac{\rho_0 c \hat{\theta} e^{j\omega t}}{3} (ka)^4 a \left[ \frac{1}{4} - \frac{3}{2}\left(\frac{\sigma}{a}\right) + \left(\frac{\sigma}{a}\right)^2 + \frac{1}{3}\left(\frac{\sigma}{a}\right)^3 \right] \sin \psi \quad (15)
 \end{aligned}$$

If this same equation is written in a form with the elliptic integrals expanded the result is

$$\begin{aligned}
p \approx & \frac{j \rho_0 c [\ell_{c.g.} - \sigma \sin \psi] \hat{\theta} e^{j\omega t}}{2} (ka) \left[ 1 - \frac{1}{4} \left( \frac{\sigma}{a} \right) - \frac{11}{32} \left( \frac{\sigma}{a} \right)^2 + \frac{1}{16} \left( \frac{\sigma}{a} \right)^3 \right. \\
& \left. - \frac{5}{256} \left( \frac{\sigma}{a} \right)^4 \right] + \frac{j \rho_0 c \hat{\theta} e^{j\omega t}}{8} (ka) a \left[ -\frac{1}{3} + 2 \left( \frac{\sigma}{a} \right) - \frac{1}{4} \left( \frac{\sigma}{a} \right)^2 - \frac{1}{4} \left( \frac{\sigma}{a} \right)^3 \right. \\
& \left. + \frac{1}{64} \left( \frac{\sigma}{a} \right)^4 \right] \sin \psi + \frac{j \rho_0 c [\ell_{c.g.} - \sigma \sin \psi] \hat{\theta} e^{j\omega t}}{2} (ka)^3 \left[ -\frac{1}{3} + \left( \frac{\sigma}{a} \right) - \frac{1}{4} \left( \frac{\sigma}{a} \right)^2 \right. \\
& \left. - \frac{1}{16} \left( \frac{\sigma}{a} \right)^3 + \frac{1}{64} \left( \frac{\sigma}{a} \right)^4 \right] - j \rho_0 c \hat{\theta} e^{j\omega t} (ka)^3 a \left[ -\frac{1}{6} + \frac{1}{4} \left( \frac{\sigma}{a} \right) - \frac{5}{8} \left( \frac{\sigma}{a} \right)^2 \right. \\
& \left. + \frac{9}{76} \left( \frac{\sigma}{a} \right)^3 - \frac{1}{180} \left( \frac{\sigma}{a} \right)^4 \right] \sin \psi + \frac{\rho_0 c [\ell_{c.g.} - \sigma \sin \psi] \hat{\theta} e^{j\omega t} (ka)^4}{3} \left[ 1 \right. \\
& \left. - \left( \frac{\sigma}{a} \right) + \frac{1}{2} \left( \frac{\sigma}{a} \right)^2 \right] + \frac{\rho_0 c \hat{\theta} e^{j\omega t}}{3} (ka)^4 a \left[ \frac{1}{4} - \frac{3}{2} \left( \frac{\sigma}{a} \right) + \left( \frac{\sigma}{a} \right)^2 + \frac{1}{3} \left( \frac{\sigma}{a} \right)^3 \right] \sin \psi
\end{aligned} \tag{76}$$

### Vertical Pressure Profiles

Equation (76) contains a special case of particular interest.

If  $\ell_{c.g.}$  is assumed to be zero this equation reduces to a disk suspended along its centerline. Under these conditions the motion is a pure rotation. Of course, a real disk under these conditions would have no restoring moment, but this could be overcome by properly weighting the lower edge of the disk. The special case of (76) which results is

$$\begin{aligned}
p = & - \frac{\rho_0 c \hat{\theta} e^{j\omega t}}{2} a \left[ j(ka) \left\{ \frac{1}{12} + \frac{1}{2} \left( \frac{\sigma}{a} \right) - \frac{3}{16} \left( \frac{\sigma}{a} \right)^2 - \frac{9}{32} \left( \frac{\sigma}{a} \right)^3 + \frac{15}{256} \left( \frac{\sigma}{a} \right)^4 \right\} \right. \\
& + j(ka)^3 \left\{ -\frac{1}{3} + \frac{1}{6} \left( \frac{\sigma}{a} \right) - \frac{1}{4} \left( \frac{\sigma}{a} \right)^2 - \frac{1}{76} \left( \frac{\sigma}{a} \right)^3 - \frac{53}{720} \left( \frac{\sigma}{a} \right)^4 \right\} \\
& \left. - (ka)^4 \left\{ \frac{1}{6} - \frac{1}{3} \left( \frac{\sigma}{a} \right) + \frac{5}{9} \left( \frac{\sigma}{a} \right)^3 \right\} \right] \sin \psi
\end{aligned} \tag{77}$$

The pressure has a simple sine dependence on  $\psi$  but a somewhat more complicated dependence on radius. The terms of (77) are plotted in figures 10, 11, and 12 in scaled form with  $\psi = \frac{\pi}{2}$ . Hence these plots represent vertical pressure profiles across the pendulum. Since no translation is present the pressure given by (27) appears along with its mirror image for  $\psi = -\frac{\pi}{2}$  as a torque on the disc which gives rise to an apparent inertia. As usual, it can be seen that at low frequencies,  $ka \ll 1$ , the higher order terms are insignificant in relation to the first term.

#### Force on the Pendulum

The effect the pressure has on the pendulum can be broken up into pure forces and torques. The force is obtained by applying equation (76) for the pressure in equation 2. In this case

$$\begin{aligned}
 F = & - \int_0^{2\pi} \int_0^a \left\{ \frac{j \rho_0 c [\ell_{c.g.} \hat{\theta} - \sigma \sin \psi]}{2} \right\} e^{j\omega t} (ka) \left[ 1 - \frac{1}{4} \left( \frac{\sigma}{a} \right) - \frac{11}{32} \left( \frac{\sigma}{a} \right)^2 + \frac{1}{16} \left( \frac{\sigma}{a} \right)^3 \right. \\
 & - \frac{5}{256} \left( \frac{\sigma}{a} \right)^4 \left. \right] + \frac{j \rho_0 c \hat{\theta} e^{j\omega t}}{8} ka^2 \left[ -\frac{1}{3} + 2 \left( \frac{\sigma}{a} \right) - \frac{1}{4} \left( \frac{\sigma}{a} \right)^2 - \frac{1}{4} \left( \frac{\sigma}{a} \right)^3 + \frac{1}{64} \left( \frac{\sigma}{a} \right)^4 \right] \sin \psi \\
 & + \frac{j \rho_0 c [\ell_{c.g.} \hat{\theta} - \sigma \sin \psi] e^{j\omega t}}{2} (ka)^3 \left[ -\frac{1}{3} + \left( \frac{\sigma}{a} \right) - \frac{1}{4} \left( \frac{\sigma}{a} \right)^2 - \frac{1}{16} \left( \frac{\sigma}{a} \right)^3 + \frac{1}{64} \left( \frac{\sigma}{a} \right)^4 \right] \\
 & - \frac{j \rho_0 c \hat{\theta} e^{j\omega t}}{4} (ka)^3 a \left[ \frac{1}{15} \left( 2 - \frac{7}{2} \left( \frac{\sigma}{a} \right)^2 - 2 \left( \frac{\sigma}{a} \right)^3 - \frac{1189}{96} \left( \frac{\sigma}{a} \right)^4 \right) \right] \sin \psi \\
 & + \rho_0 c [\ell_{c.g.} \hat{\theta} - \sigma \sin \psi] e^{j\omega t} (ka)^4 \left[ 1 - \left( \frac{\sigma}{a} \right) + \frac{1}{2} \left( \frac{\sigma}{a} \right)^2 \right] \\
 & + \frac{\rho_0 c \hat{\theta} e^{j\omega t}}{3} (ka)^4 a \left[ \frac{1}{4} - \frac{3}{2} \left( \frac{\sigma}{a} \right) + \left( \frac{\sigma}{a} \right)^2 + \frac{1}{3} \left( \frac{\sigma}{a} \right)^3 \right] \sin \psi \} \sigma \, d\sigma \, d\psi \quad (78)
 \end{aligned}$$

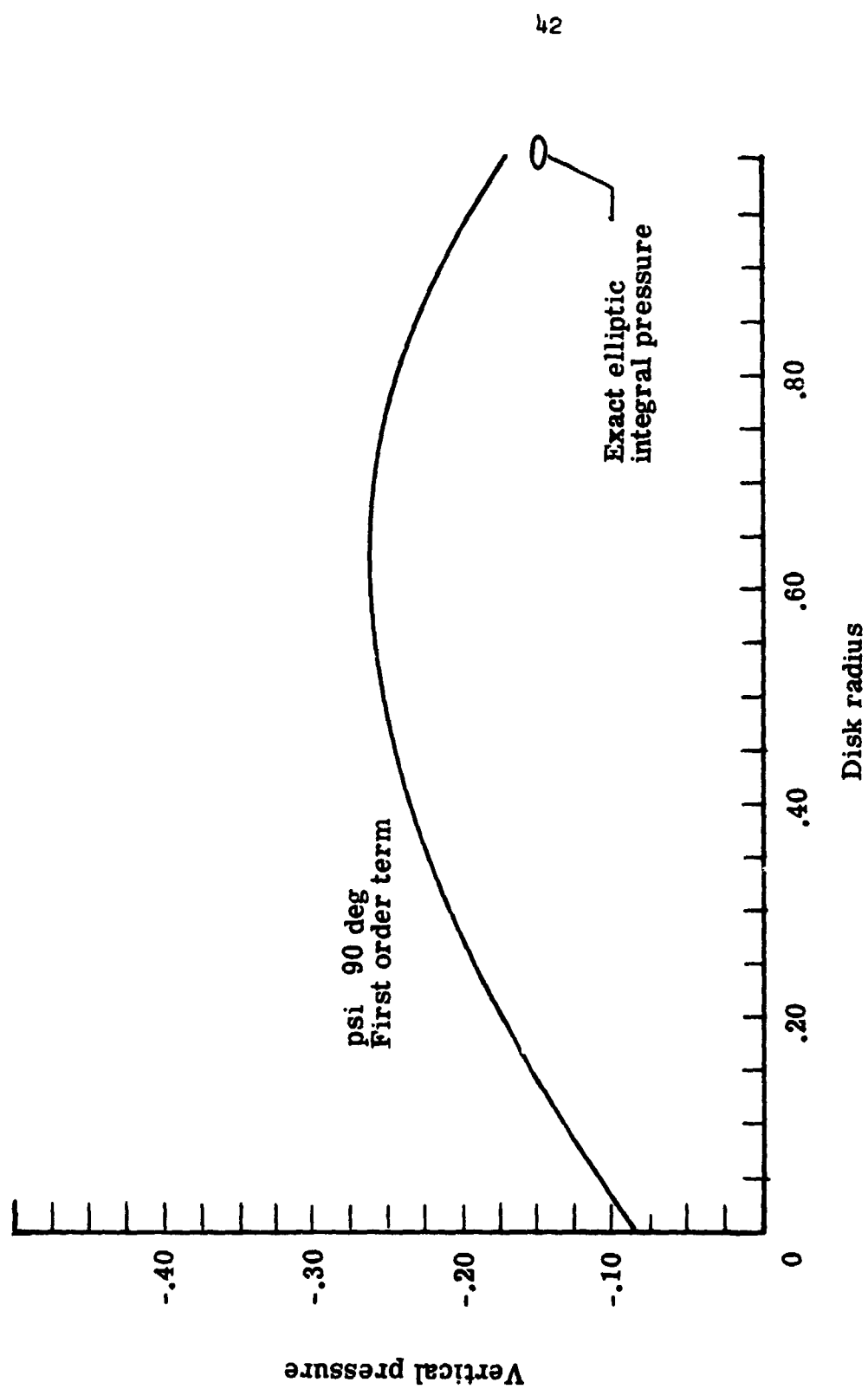


Figure 10.- Vertical pressure across a disk in rotation due to the first attached inertia term as a function of scaled disk radius.

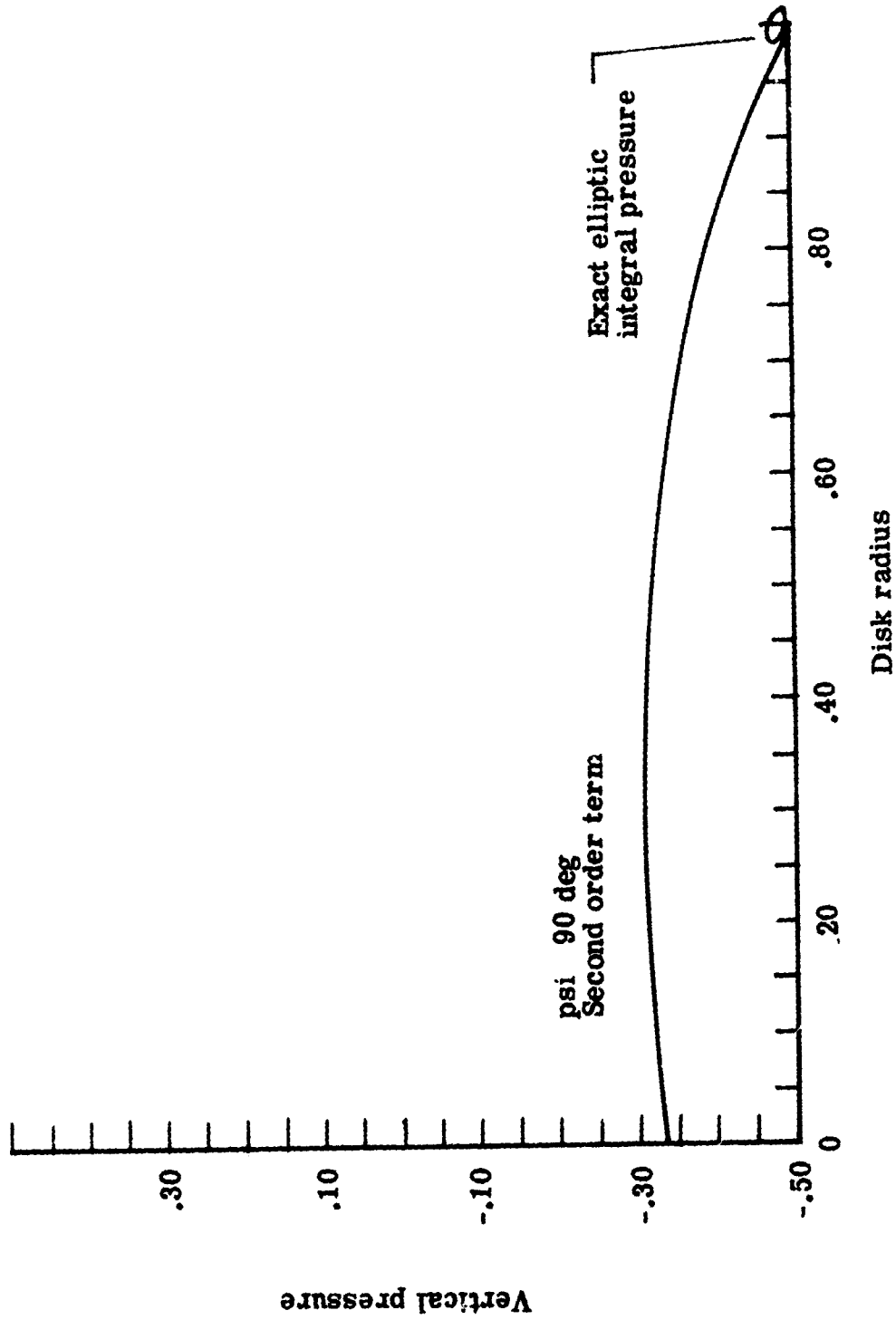


Figure 11.- Vertical pressure across a disk in rotation due to the second attached inertia term as a function of scaled disk radius.

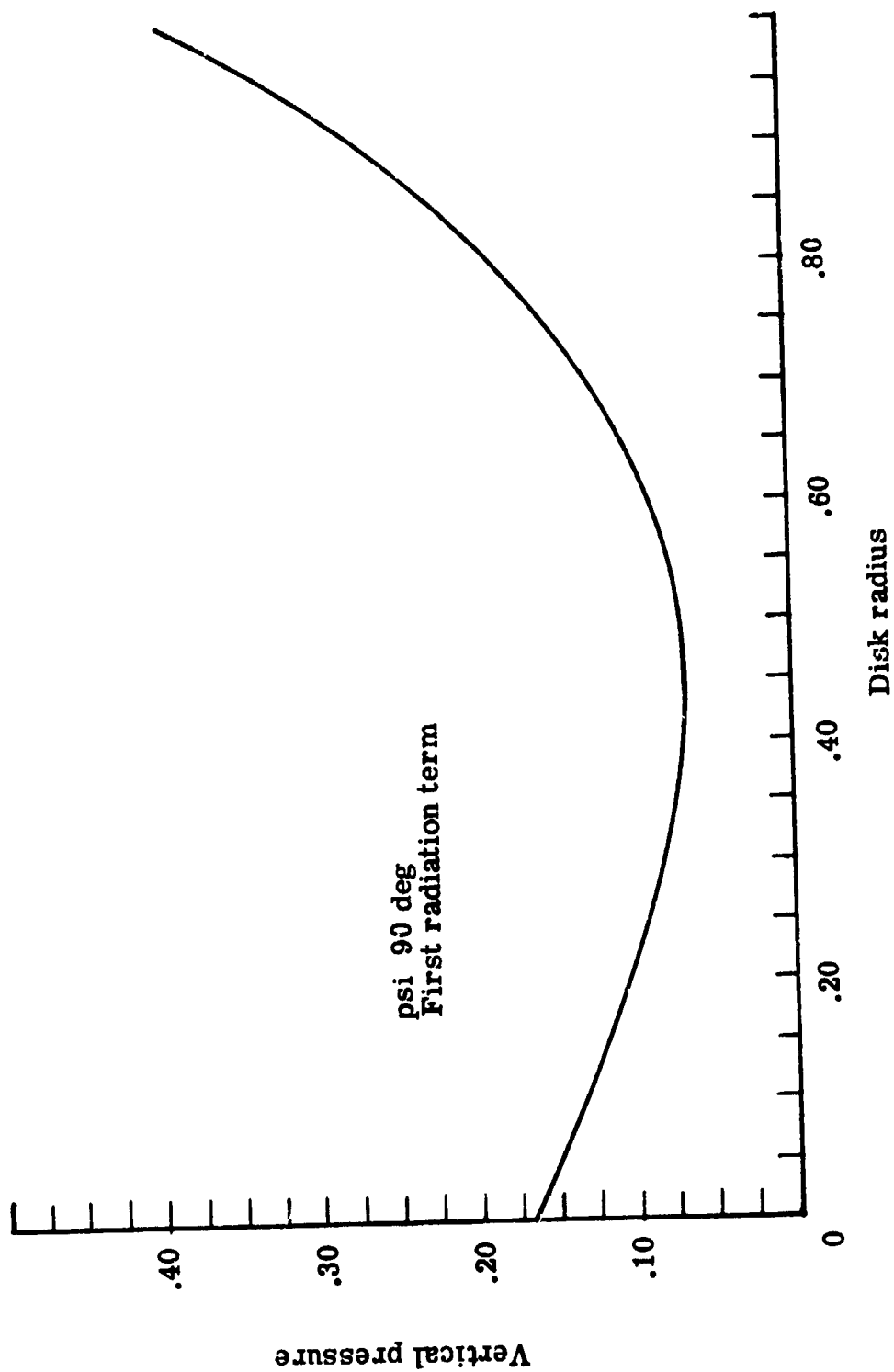


Figure 12.- Vertical pressure due to rotation for the first non-vanishing radiation term as a function of scaled disk radius.

Since the order of integration is immaterial in this instance, it is advantageous to integrate over  $\psi$  first. In this case all terms containing  $\sin \psi$  drop out; and the force equation becomes

$$\begin{aligned}
 F = & -\rho_0 c \ell_{c.g.} \hat{\theta} e^{j\omega t} 2\pi \int_0^a \left\{ \frac{j(ka)}{2} \left[ 1 - \frac{1}{4}\left(\frac{\sigma}{a}\right) - \frac{11}{32}\left(\frac{\sigma}{a}\right)^2 + \frac{1}{16}\left(\frac{\sigma}{a}\right)^3 - \frac{5}{256}\left(\frac{\sigma}{a}\right)^4 \right] \right. \\
 & + \frac{j(ka)^3}{2} \left[ -\frac{1}{3} + \left(\frac{\sigma}{a}\right) - \frac{1}{4}\left(\frac{\sigma}{a}\right)^2 - \frac{1}{16}\left(\frac{\sigma}{a}\right)^3 + \frac{1}{64}\left(\frac{\sigma}{a}\right)^4 \right] \\
 & \left. + \frac{(ka)^4}{3} \left[ 1 - \left(\frac{\sigma}{a}\right) + \frac{1}{2}\left(\frac{\sigma}{a}\right)^2 \right] \right\} \sigma d\sigma
 \end{aligned} \tag{79}$$

This is seen to be identical to equation (35) except that  $\hat{u}$  is replaced by  $\ell_{c.g.} \hat{\theta}$ . Thus with the second integration carried out, the force on one face is found to be

$$F = -\rho_0 c \ell_{c.g.} \hat{\theta} e^{j\omega t} a^2 \{ j[1.0681(ka) + 0.2962(ka)^3] + [0.6109(ka)^4] \} \tag{80}$$

The force on both faces is twice this, or

$$F = -\rho_0 c \ell_{c.g.} \hat{\theta} e^{j\omega t} a^2 \{ j[2.1362(ka) + 0.5924(ka)^3] + [1.2218(ka)^4] \} \tag{81}$$

#### Attached Mass of the Pendulum

Since the acceleration of the pendulum is given by

$$\frac{dv}{dt} = -j\omega l_{c.g.} \hat{\theta} e^{j\omega t} \quad (82)$$

the attached mass is found by dividing the first term of (81) by (82)

$$m_a = 2.1362 \rho_0 a^3 \quad (83)$$

This is exactly the same as was found for a simple disk in translation.

#### Torque on the Pendulum Disk

The torque equation is obtained by integrating the pressure equation over the disk with the appropriate moment arm. If the moment arm is selected from the centerline of the disk

$$\tau = - \int_0^{2\pi} \int_0^a p(\sigma \sin \psi) \sigma d\sigma d\psi \quad (84)$$

or, applying equation (76) for pressure



$$\begin{aligned}
\tau = - \int_0^{2\pi} \int_0^a & \left\{ \frac{j \rho_0 c [\ell_{c.g.} - \sigma \sin \psi] \hat{\theta} e^{j\omega t}}{2} (ka)^3 \left[ 1 - \frac{1}{4} \left( \frac{\sigma}{a} \right) - \frac{11}{32} \left( \frac{\sigma}{a} \right)^2 + \frac{1}{16} \left( \frac{\sigma}{a} \right)^3 \right. \right. \\
& - \frac{5}{256} \left( \frac{\sigma}{a} \right)^4 + \frac{j \rho_0 c \hat{\theta} e^{j\omega t}}{8} (ka) a \left[ -\frac{1}{3} + 2 \left( \frac{\sigma}{a} \right) - \frac{1}{4} \left( \frac{\sigma}{a} \right)^2 - \frac{1}{4} \left( \frac{\sigma}{a} \right)^3 + \frac{1}{64} \left( \frac{\sigma}{a} \right)^4 \right] \sin \psi \\
& + \frac{j \rho_0 c [\ell_{c.g.} - \sigma \sin \psi] \hat{\theta} e^{j\omega t} (ka)^3}{2} \left[ -\frac{1}{3} + \left( \frac{\sigma}{a} \right) - \frac{1}{4} \left( \frac{\sigma}{a} \right)^2 - \frac{1}{16} \left( \frac{\sigma}{a} \right)^3 + \frac{1}{64} \left( \frac{\sigma}{a} \right)^4 \right] \\
& - j \rho_0 c \hat{\theta} e^{j\omega t} (ka)^3 a \left[ -\frac{1}{6} + \frac{1}{4} \left( \frac{\sigma}{a} \right) - \frac{5}{8} \left( \frac{\sigma}{a} \right)^2 + \frac{9}{76} \left( \frac{\sigma}{a} \right)^3 - \frac{1}{180} \left( \frac{\sigma}{a} \right)^4 \right] \sin \psi \\
& + \frac{\rho_0 c [\ell_{c.g.} - \sigma \sin \psi]}{3} \hat{\theta} e^{j\omega t} (ka)^4 \left[ 1 - \left( \frac{\sigma}{a} \right) + \frac{1}{2} \left( \frac{\sigma}{a} \right)^2 \right] \\
& \left. + \frac{\rho_0 c \hat{\theta} e^{j\omega t}}{3} (ka)^4 a \left[ \frac{1}{4} - \frac{3}{2} \left( \frac{\sigma}{a} \right) + \left( \frac{\sigma}{a} \right)^2 + \frac{1}{3} \left( \frac{\sigma}{a} \right)^3 \right] \sin \psi \right\} (\sigma \sin \psi) \sigma d\sigma d\psi
\end{aligned} \tag{85}$$

As before, since the order of integration is immaterial, the best policy is to integrate over  $\psi$  first. In this instance terms containing  $\ell_{c.g.}$  drop out of the integral and one has

$$\begin{aligned}
\tau = -\pi \int_0^a \{ & -j \rho_0 c \hat{\theta} e^{j\omega t} \frac{(ka)}{2} [\sigma^3 - \frac{1}{4} \frac{\sigma^4}{a} - \frac{11}{32} \frac{\sigma^5}{a^2} + \frac{1}{16} \frac{\sigma^6}{a^3} - \frac{5}{256} \frac{\sigma^7}{a^4}] \\
& + \frac{j \rho_0 c \hat{\theta} e^{j\omega t}}{8} (ka) a [-\frac{\sigma^2}{3} + \frac{2\sigma^3}{a} - \frac{1}{4} \frac{\sigma^4}{a^2} - \frac{1}{4} \frac{\sigma^5}{a^3} + \frac{1}{64} \frac{\sigma^6}{a^4}] \\
& - \frac{j \rho_0 c \hat{\theta} e^{j\omega t}}{2} (ka)^3 [-\frac{1}{3} \sigma^3 + \frac{\sigma^4}{a} - \frac{1}{4} \frac{\sigma^5}{a^2} - \frac{1}{16} \frac{\sigma^6}{a^3} + \frac{1}{64} \frac{\sigma^7}{a^4}] \\
& - j \rho_0 c \hat{\theta} e^{j\omega t} (ka)^3 a [-\frac{1}{6} \sigma^2 + \frac{1}{4} \frac{\sigma^3}{a} - \frac{5}{8} \frac{\sigma^4}{a^2} + \frac{9}{76} \frac{\sigma^5}{a^3} - \frac{1}{180} \frac{\sigma^6}{a^4}] \\
& - \frac{\rho_0 c \hat{\theta} e^{j\omega t}}{3} (ka)^4 [\sigma^3 - \frac{\sigma^4}{a} + \frac{1}{2} \frac{\sigma^5}{a^2}] \\
& + \frac{\rho_0 c \hat{\theta} e^{j\omega t}}{3} (ka)^4 a [\frac{1}{4} \sigma^2 - \frac{3}{2} \frac{\sigma^3}{a} + \frac{\sigma^4}{a^2} + \frac{1}{3} \frac{\sigma^5}{a^3}] \} d\sigma \quad (86)
\end{aligned}$$

If this equation is integrated again and simplified the torque due to the pressure acting on one face of the disk is

$$\tau = \pi \rho_0 c \hat{\theta} e^{j\omega t} ka^5 \{ j[0.0372 - 0.1046(ka)^2] + [0.0565(ka)^3] \} \quad (87)$$

The total torque due to the pressure on both faces is twice this, or

$$\tau = \pi \rho_0 c \hat{\theta} e^{j\omega t} ka^5 \{ j[0.0744 - 0.2092(ka)^2] + [0.1130(ka)^3] \} \quad (88)$$

#### Apparent Inertia

The apparent inertia is given by the reaction to the torque.

In the case of a simple disk in translation there is no torque and,

therefore, no apparent inertia. However, in the case of a pendulum

$$\tau = I_a \frac{d\dot{\theta}}{dt} \quad (89)$$

and since

$$\frac{d\dot{\theta}}{dt} = j \omega \hat{\theta} e^{j\omega t} \quad (90)$$

The real part of  $I_a$ , to first order, is given by

$$I_a = \rho_0 \left[ \frac{4}{3} \pi a^3 \right] a^2 [0.0279] \quad (91)$$

or

$$I_a = 0.1169 \rho_0 a^5$$

The apparent inertia due to both faces is twice this, or

$$I_a = \rho_0 \left[ \frac{4}{3} \pi a^3 \right] a^2 [0.0558] \quad (92)$$

or

$$I_a = 0.2338 \rho_0 a^5 \quad (93)$$

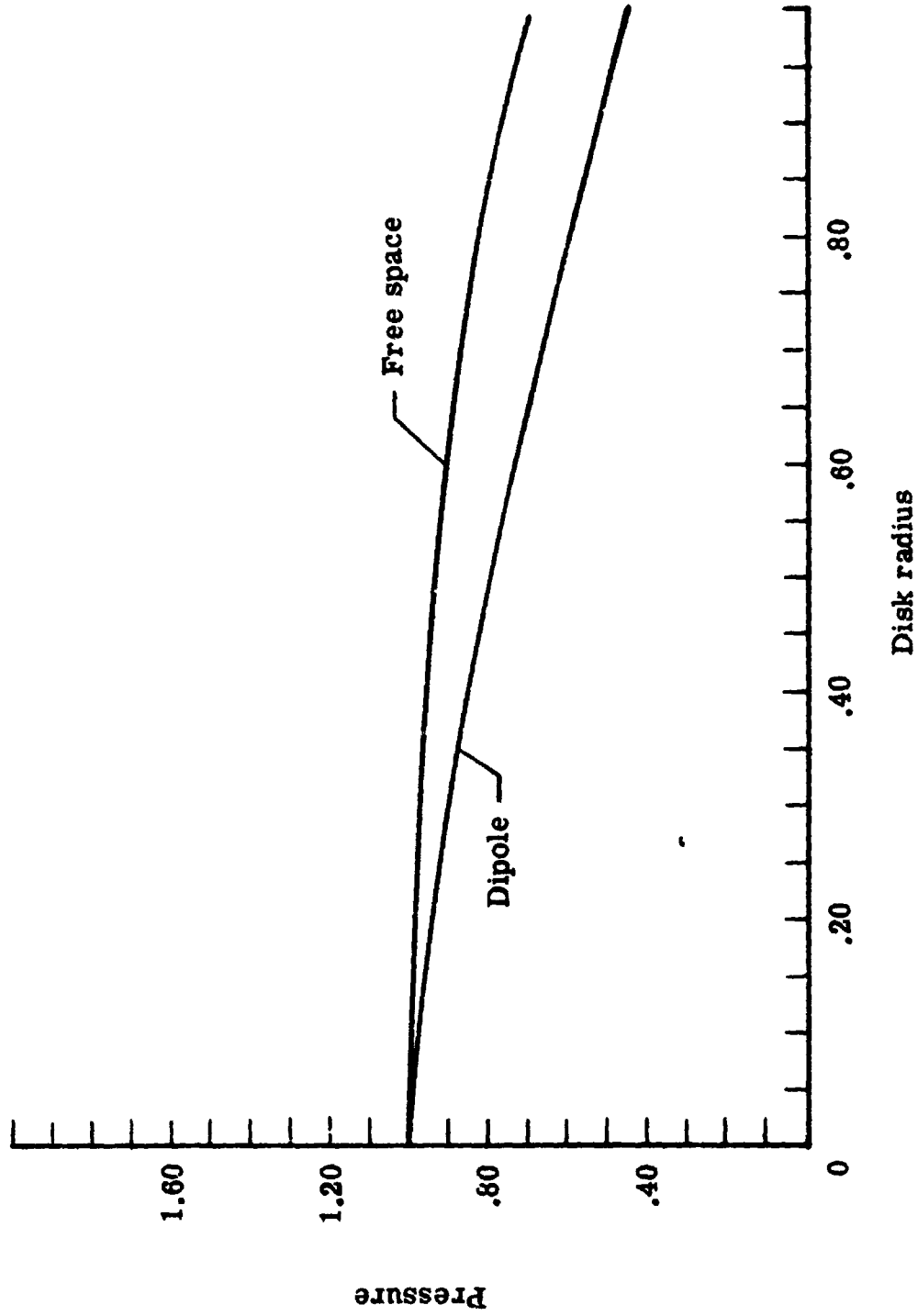


Figure 13.- Comparison of the first attached mass pressure term for a dipole with the free space Green's function for the same disk as a function of scaled disk radius.

## DISCUSSION OF RESULTS

The results of the present investigation can be compared with known results from similar investigations in several instances. First, the impedance from a distributed dipole is somewhat larger than that which would be calculated from a point dipole of equivalent strength as was shown in equations (38) and (39). The reason for this is that a distributed disk dipole acts to some extent as its own baffle. That is, pressure waves originating in the center of the disk have to travel farther to get around the edge than is the case for a point dipole. In doing this the sound is attenuated by  $\frac{1}{r}$  and thus cancellation between the out of phase components is less pronounced. If this is the case, then an upper limiting case should be available in the free space Green's function model since this is equivalent to an infinite baffle with no phase cancellation. A comparison of the dominant first order pressure along a radius between the free space model given in Appendix A and the dipole model is given in figure 13. It can be seen that the pressure due to the dipole drops off more rapidly all along the radius. If more terms had been carried in the expansions the dipole pressure would be exactly zero at the edge of the disk. That is, of course, not true for the free space model since zero boundary conditions were not specified in the Green's function. A similar comparison of pressures is given in figure 14 for pure rotation. Here the pressure builds more rapidly and to a higher value before falling off near the edge. The result is that a dipole model predicts less torque since the

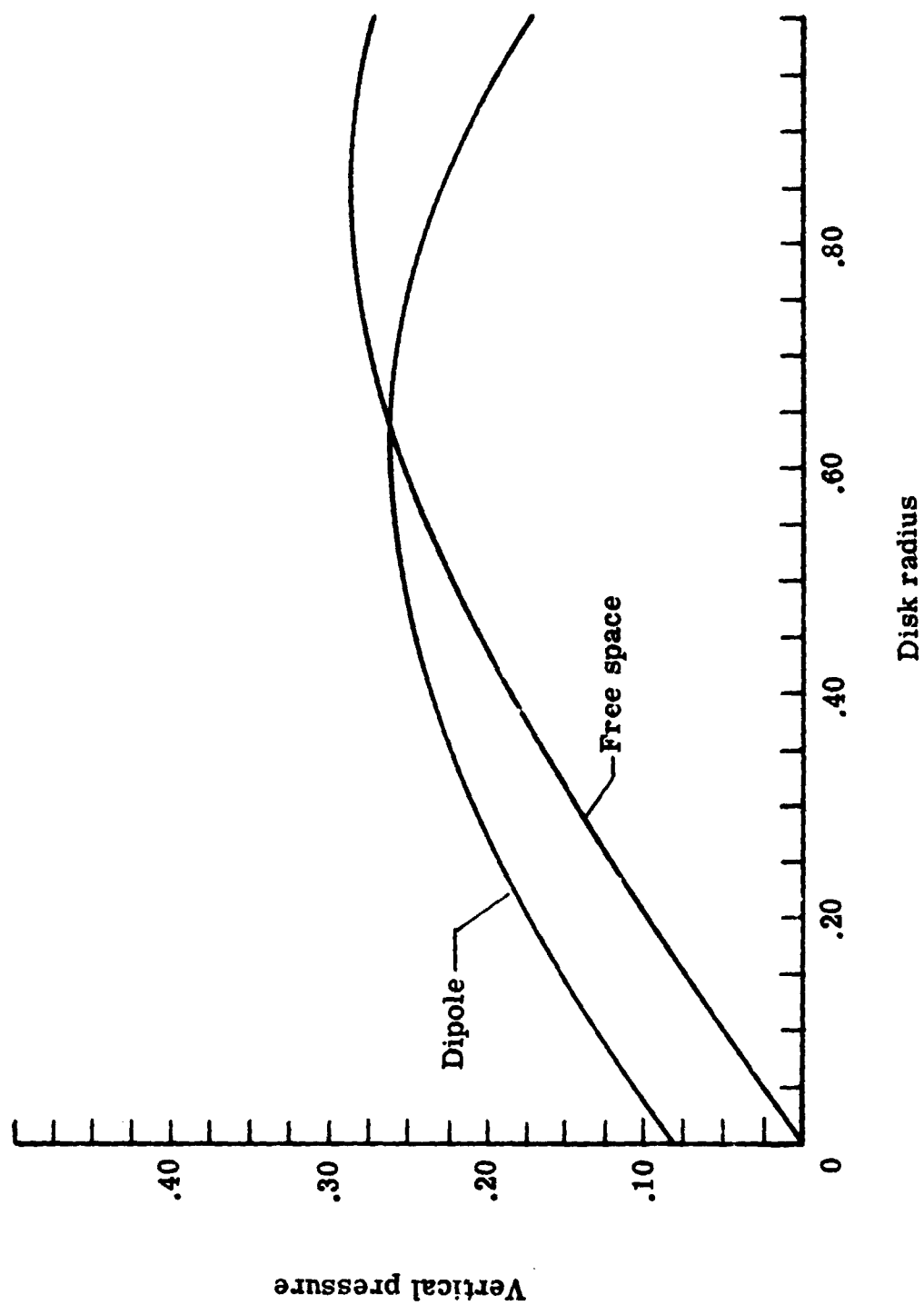


Figure 14.- Comparison of the vertical pressures between a dipole model and the corresponding free space term as a function of scaled disk radius.

maximum pressure is applied near the center of the disk with a shorter effective lever arm.

The results of the present analysis can also be compared with experiment. Table I (modified from reference 13) gives experimental values of  $(ka)$  for the pendulum experiment done in this reference. It is apparent that for all measurements  $ka < .006$ . Hence the pendulum of this experiment should be described acoustically to sufficient accuracy by the first order terms developed in the present paper. No direct acoustic pressures were measured in this experiment. However, the attached mass and inertia were calculated. These parameters are given in Table II. In addition accepted values of these parameters from hydrodynamics are given. In this latter instance (reference 14) the attached mass is calculated from the kinetic energy and free stream velocity of air flowing past a disk for a non-viscous fluid.

The conclusion to be drawn from this table is that the dipole model gives the lowest attached mass value of all models.

Since the experimental values are higher than the values of any of the theoretical models it seems likely that additional air is dragged along with the pendulum which remains unexplained. Some of this mass is likely to lie beyond the edge of the disk where only the model given by Lamb is rigorously applicable. Some support for this idea is to be found in that the free field model of Appendix A also gives higher values, and it was pointed out previously that this model does not presuppose zero pressure at the edge of the disk. An additional physical effect not accounted for by the present theory is

the possibility of wake formation behind the disk as it swings. Since the period of the pendulum is not measurably different for amplitudes between approximately thirty degrees and the limits of visual perception this is not considered significant. However, the possibility of vortex shedding in a wake is not to be completely discounted. For the same reason direct friction effects on the pendulum are not thought to be significant. However, it is quite possible that attenuation of the pressure wave as it travels around the edge of the disk occurs. This could be of two forms. Either the wave shape could spread so that peak amplitude is lowered, or the wave could fail to diffract around the edge of the disk for reasons which are not at present explained. It is not thought that node formation is a factor in the present situation since the effect would be to lower further the already low attached mass values (reference 2).



## CONCLUSIONS

It is feasible to describe the motion of a disk pendulum by breaking the motion up into pure rotation and pure translation and treating the surface as a distributed acoustic oscillator operating at very low frequency. However, the dipole model analysis of the present paper gives lower values for attached mass and attached inertia at low frequency than either the hydrodynamic model of Lamb or previous experiment undertaken by the author. The conclusion is reached that, although this model is reasonable from a qualitative standpoint, it is only marginally adequate in its present form to explain the experimental results. The model is also not in satisfactory numerical agreement with hydrodynamic theory.

## SUGGESTIONS FOR SUBSEQUENT RESEARCH

Since the present analysis gives a theoretical value for pressure at any point on a disk pendulum it would be useful to know actual experimental relative pressure profiles across a large disk pendulum. The scaled nature of the theory suggests that it would not be necessary to obtain absolute numerical results since pressure relative to pressure at the center of the disk would be satisfactory; thus a simple experiment could be devised. It would also be useful to have wind tunnel pressure profile data for a simple disk in a very low speed air stream at various angles of attack. Wind tunnel data at the extremely low speeds suggested have not been readily available in the past. This information should be useful in determining the effects of wake and vortex action if present and should also give an idea of the air flow pattern across the leading and trailing faces of the disk.

It is also suggested that a study be made of the effect of the test facility walls as the necessity for working in a finite chamber, the wavelength is one to three kilometers in reference 13, may impose additional boundary conditions not taken into account in the present study. In addition this problem is amendable to scaling as has been done in the present paper by means of the factor  $ka$ . Thus it should be possible to obtain results at higher frequencies for shorter wavelengths as, for example, by loudspeaker experiments.

TABLE I.- SUMMARY OF EXPERIMENTAL DATA USING TWO DATA-REJECTION CRITERIA

[Disk radius = 0.9525 meter]

$\lambda$ meters	Speed of sound c, m/sec	$k$ $\left(\frac{2\pi}{pc}\right)$	$ka$	Mass, m, kg	Density, $\rho$ , kg/m <sup>3</sup>	$l_{cg}$ , m	2 $\sigma$ data-rejection criterion (a)		1.5 $\sigma$ data-rejection criterion (b)	
							Mean period, sec	Standard deviation, $\sigma$ , sec	Mean period, sec	Standard deviation, $\sigma$ , sec
1098.46	340.690	0.00572	0.00545	4.42	0.00161	0.00	3.220	0.000	3.220	0.000
1215.32	340.237	.00572	.00545	4.42	.0339	weighted slug	3.230	.000	3.230	.000
1126.02	341.395	.00558	.00532	4.42	.287	of mass	3.297	.005	3.297	.005
1142.48	339.218	.00549	.00523	4.42	.609	0.850 kg	3.373	.010	3.375	.010
1167.89	339.598	.00538	.00512	4.42	.875	0.902 m from	3.436	.009	3.440	.000
1190.00	338.429	.00528	.00503	4.42	1.24	c.g.	3.517	.018	3.517	.018
1521.35	351.790	0.00413	0.00393	4.21	0.00150	4.57	4.323	0.005	4.323	0.005
1468.03	335.327	.00428	.00408	4.21	.0349	4.57	4.375	.005	4.375	.005
1602.85	341.395	.00392	.00373	4.21	.287	4.57	4.690	.000	4.690	.000
1702.76	336.739	.00369	.00351	4.21	.618	4.57	5.060	.000	5.060	.000
1795.20	337.483	.00350	.00333	4.21	.886	4.57	5.307	.010	5.307	.010
1886.84	338.429	.00333	.00317	4.21	1.24	4.57	5.582	.016	5.582	.016
1712.03	335.520	0.00367	0.00350	4.21	0.00166	6.40	5.100	0.000	5.100	0.000
1745.33	337.755	.00360	.00343	4.21	.0344	6.40	5.168	.008	5.165	.005
1875.58	339.624	.00335	.00319	4.21	.290	6.40	5.524	.008	5.524	.005
2046.64	341.185	.00307	.00292	4.21	.602	6.40	5.990	.018	5.987	.010
2137.13	340.572	.00294	.00280	4.21	.870	6.40	6.285	.015	6.290	.011
2284.79	341.192	.00275	.00262	4.21	1.22	6.40	6.691	.010	6.691	.010
1945.26	340.690	0.00323	0.00308	4.21	0.00161	8.04	5.705	0.005	5.706	0.005
1969.65	340.740	.00319	.00304	4.21	.0338	8.04	5.775	.016	5.774	.012
2115.55	340.212	.00297	.00283	4.21	.289	8.04	6.208	.016	6.214	.011
2293.13	341.468	.00274	.00261	4.21	.601	8.04	6.705	.037	6.705	.037
2425.94	339.598	.00259	.00247	4.21	.875	8.04	7.140	.016	7.140	.016
2575.08	338.429	.00244	.00232	4.21	1.24	8.04	7.594	.019	7.600	.000
2073.66	340.689	0.00303	0.00289	4.21	0.00161	9.14	6.088	0.008	6.089	0.007
2094.40	340.237	.00300	.00286	4.21	.0339	9.14	6.159	.008	6.159	.008
2268.30	341.395	.00277	.00264	4.21	.287	9.14	6.640	.016	6.635	.013
2260.14	339.041	.00278	.00265	4.21	.291	9.14	6.668	.020	6.655	.005
2444.82	339.218	.00257	.00245	4.21	.609	9.14	7.208	.018	7.208	.010
2435.34	337.834	.00258	.00246	4.21	.614	9.14	7.195	.018	7.200	.000
2585.67	339.792	.00243	.00231	4.21	.874	9.14	7.596	.044	7.608	.010
2575.08	338.247	.00244	.00232	4.21	.882	9.14	7.602	.016	7.602	.016
2743.75	338.429	.00229	.00218	4.21	1.24	9.14	8.099	.015	8.099	.015

(a) Data are rejected if the absolute value of the deviation exceeds two times the standard deviation.

(b) Data are rejected if the absolute value of the deviation exceeds 1.5 times the standard deviation.

TABLE II.- SUMMARY OF APPARENT MASS AND APPARENT INERTIA  
RESULTS OF SEVERAL INVESTIGATIONS

	$m_a$	$I_a$
Lamb (Hydrodynamic)	$2.667 \rho_0 a^3$	$0.355 \rho_0 a^5$
Dunning (Dipole)	$2.136 \rho_0 a^3$	$0.234 \rho_0 a^5$
Dunning (Free space)	$2.699 \rho_0 a^3$	$0.279 \rho_0 a^5$
Dunning (Experimental)	$(2.9269 \pm 0.1195) \rho_0 a^3$	$(0.3933 \pm 0.0127) \rho_0 a^5$

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## APPENDIX A

### Monopole Disk Model

It is useful for comparison purposes to determine the results for a pendulum using only the free space Green's function. Under those conditions the pendulum would be so arranged that pressure on one face has no effect on the other face. Such would be the case, for instance, for an idealized pendulum with the faces isolated from each other by an infinite baffle. Since the translational part of the normal pendulum analysis corresponds so closely with a distributed dipole disk, it stands to reason that a baffled pendulum would correspond closely to the well known example of a distributed monopole as, for instance, a loudspeaker in a baffle. The pressure distribution for this configuration is well known (reference 2).

A starting point is to be had with equation 7.

$$p_1 = \int_0^a \int_0^{2\pi} \frac{2\pi a \left[ \frac{a}{a} \cos \phi + \left( 1 - \left( \frac{a}{a} \right)^2 \sin^2 \phi \right)^{1/2} \right] \rho_0 c k \hat{u} e^{j(\omega t - k r)}}{4\pi} dr d\phi \quad (7)$$

Since the first radiation term is not self-cancelling in this case it is sufficient to retain only the first order power of  $k$ .

$$p_i = \int_0^a \int_0^{2\pi} \frac{j \rho_0 c k \hat{u} e^{j(\omega t)} \left[ \frac{\sigma}{a} \cos \phi + \left(1 - \left(\frac{\sigma}{a}\right)^2 \sin^2 \phi\right)^{1/2} \right]}{4\pi} [1 - jkr] dr d\phi \quad (A-1)$$

The first integral of this is found readily and is

$$p_i = \frac{\rho_0 c k \hat{u} e^{j\omega t}}{4\pi} \int_0^{2\pi} \left\{ \frac{ka^2}{2} [1 + \left(\frac{\sigma}{a}\right)^2 (\cos^2 \phi - \sin^2 \phi) + 2\left(\frac{\sigma}{a}\right) \cos \phi (1 - \left(\frac{\sigma}{a}\right)^2 \sin^2 \phi)^{1/2}] + a \left[ \frac{\sigma}{a} \cos \phi + \left(1 - \left(\frac{\sigma}{a}\right)^2 \sin^2 \phi\right)^{1/2} \right] \right\} d\phi \quad (A-2)$$

The second term in the first bracket integrates to zero because of symmetry. Hence the pressure is

$$p_i = \frac{\rho_0 c \hat{u} e^{j\omega t}}{4} [(ka)^2 + \frac{4j(ka)}{\pi} E(\frac{\pi}{2}, \frac{\sigma}{a})] \quad (A-3)$$

When the elliptic integral is expanded by means of (24) this becomes

$$p_i \approx \frac{\rho_0 c \hat{u} e^{j\omega t}}{4} [(ka)^2 + 2j ka \{1 - \frac{1}{4}\left(\frac{\sigma}{a}\right)^2 - \frac{3}{64}\left(\frac{\sigma}{a}\right)^4 + \dots\}] \quad (A-4)$$

This result is exact to the first power of  $(ka)$ .

The force on the disk is found by integrating over the surface

$$F = - \frac{\rho_0 c \hat{u} e^{j\omega t}}{4} \int_0^{2\pi} \int_0^a [(ka)^2 + 2j ka \{1 - \frac{1}{4}\left(\frac{\sigma}{a}\right)^2 - \frac{3}{64}\left(\frac{\sigma}{a}\right)^4\}] \sigma d\sigma d\psi \quad (A-5)$$



where  $\psi$  is measured about the center of the disk. This can be integrated directly to give

$$F = - \frac{\pi \rho_0 c \hat{u} e^{j\omega t}}{2} \left[ \frac{k^2 a^4}{2} + j \left( \frac{55}{64} \right) k a^3 \right] \quad (A-6)$$

The imaginary part of this equation can be used to calculate the apparent mass. With  $k = \frac{\omega}{c}$

$$F_{\text{imaginary}} = \left( \frac{55}{64} \right) \pi \rho_0 a^3 [-j \omega \hat{u} e^{j\omega t}] \quad (A-7)$$

The term in brackets is the acceleration of the disk. Hence the apparent mass is

$$m_a = 2.699 \rho_0 a^3 \quad (A-8)$$

Equation (7) can be transformed to the pendulum form by inserting (51a) in place of  $\hat{u}$ . Then equation (A-4) still holds if  $p_I$  with  $(l_{c.g.} - \sigma \sin \psi) \hat{\theta}$  instead of  $\hat{u}$  and an additional term necessary to complete the description. This term is

$$p_{II} = - \frac{j \rho_0 c k \hat{\theta} e^{j\omega t}}{4\pi} \int_0^{2\pi} \int_0^a r e^{-jkr} \sin(\phi - \psi) dr d\phi \quad (A-9)$$

$$2\pi a \left[ \frac{\sigma}{a} \cos \phi + \left( 1 - \left( \frac{\sigma}{a} \right)^2 \sin^2 \phi \right)^{1/2} \right]$$

Expanding the exponential and keeping the first order terms as before this equation integrates over  $r$  to the form

$$\begin{aligned}
P_{II} = & - \frac{j \rho_0 c k \hat{\theta} e^{j\omega t}}{4\pi} \int_0^{2\pi} \left[ \frac{ka^3}{3} \left\{ \left(\frac{\sigma}{a}\right)^3 \cos^3 \phi + 3\left(\frac{\sigma}{a}\right)^2 \cos^2 \phi \left(1 - \left(\frac{\sigma}{a}\right)^2 \sin^2 \phi\right)^{1/2} \right. \right. \\
& + 3\left(\frac{\sigma}{a}\right) \cos \phi \left(1 - \left(\frac{\sigma}{a}\right)^2 \sin^2 \phi\right) \\
& + \left. \left. \left(1 - \left(\frac{\sigma}{a}\right)^2 \sin^2 \phi\right)^{3/2} \right\} + \frac{ka^2}{2} \left\{ \left(\frac{\sigma}{a}\right)^2 (\cos^2 \phi - \sin^2 \phi) \right. \right. \\
& + \left. \left. 1 + 2\left(\frac{\sigma}{a}\right) \cos \phi \left(1 - \left(\frac{\sigma}{a}\right)^2 \sin^2 \phi\right) \right\} \right] \sin(\phi - \psi) d\phi \quad (A-10)
\end{aligned}$$

as before, making use of the identity

$$\sin(\phi - \psi) = \sin \phi \cos \psi - \cos \phi \sin \psi$$

and integrating, most terms drop out; and the result is

$$\begin{aligned}
P_{II} = & \frac{\rho_0 c \hat{\theta} e^{j\omega t}}{4\pi} \left[ \pi(ka)^2 a \left(\frac{\sigma}{a}\right) + j \frac{4}{3} (ka) a \left\{ \frac{1 + \left(\frac{\sigma}{a}\right)^2}{\left(\frac{\sigma}{a}\right)} E\left(\frac{\pi}{2}, \frac{\sigma}{a}\right) \right. \right. \\
& \left. \left. - \frac{1 - \left(\frac{\sigma}{a}\right)^2}{\left(\frac{\sigma}{a}\right)} K\left(\frac{\pi}{2}, \frac{\sigma}{a}\right) \right\} \right] \sin \psi \quad (A-11)
\end{aligned}$$

With the elliptic integrals expanded by (23) and (24) this becomes

$$P_{II} = \frac{\rho_0 c \hat{\theta} e^{j\omega t}}{4} \left(\frac{\sigma}{a}\right) \left[ (ka)^2 a + j(ka) a \left(1 - \frac{1}{5} \left(\frac{\sigma}{a}\right)^2 - \frac{1}{64} \left(\frac{\sigma}{a}\right)^4 \right) \right] \quad (A-12)$$

$P_I$  and  $P_{II}$  are then added to get the total pressure

$$\begin{aligned}
p_i &= \frac{\rho_0 c \hat{\theta} e^{j\omega t}}{4} \{ l_{c.g.} (ka)^2 + j(ka) [ l_{c.g.} \{ 2 - \frac{1}{2}(\frac{\sigma}{a})^2 - \frac{3}{32}(\frac{\sigma}{a})^4 \} \\
\text{free space} & - \sigma \sin \psi \{ 1 - \frac{3}{8}(\frac{\sigma}{a})^2 - \frac{5}{64}(\frac{\sigma}{a})^4 \} ] \} \quad (A-13)
\end{aligned}$$

The torque equation is obtained by integrating the pressure equation over the disk with the appropriate moment arm

$$\begin{aligned}
\tau &= \frac{\rho_0 c \hat{\theta} e^{j\omega t}}{4} \int_0^{2\pi} \int_0^a \{ (ka)^2 l_{c.g.} + j(ka) [ l_{c.g.} (2 - \frac{1}{2}(\frac{\sigma}{a})^2 + \frac{3}{32}(\frac{\sigma}{a})^4 ) \\
& - \sigma \sin \psi (1 - \frac{3}{8}(\frac{\sigma}{a})^2 - \frac{5}{64}(\frac{\sigma}{a})^4 ) ] \} \sigma \sin \psi \sigma d\sigma d\psi \quad (A-14)
\end{aligned}$$

Integration over  $\psi$  first gives zero for all terms containing  $l_{c.g.}$ . Thus the result is

$$\begin{aligned}
\tau &= - \frac{\pi \rho_0 c \hat{\theta} e^{j\omega t}}{4} \int_0^a j(ka) \{ \sigma^3 - \frac{3}{8} \frac{\sigma^5}{a^2} - \frac{5}{64} \frac{\sigma^7}{a^4} \} d\sigma \\
&= - j [ (\frac{\pi}{4}) (\frac{91}{512}) \rho_0 a^5 ] \omega \hat{\theta} e^{j\omega t} \quad (A-15)
\end{aligned}$$

Then dividing by (90) the attached inertia becomes

$$I_a = 0.2792 \rho_0 a^5 \quad (A-16)$$

## APPENDIX B

### SOLUTION TO THE EQUATION OF MOTION OF A PENDULUM WITH AERODYNAMIC DRAG

A perturbation solution to equation (50) is given in this appendix. This solution is due to E. M. McDaid and is given here for the sake of completeness.

Equation (50) can be written

$$\ddot{\theta} + \frac{1}{2} \frac{\rho(l_{\text{aero}})^3 s c_D}{I} \dot{\theta} |\dot{\theta}| + \frac{m}{I} \frac{g l_{\text{c.g.}}}{I} \theta = 0 \quad (\text{B-1})$$

Now let

$$\epsilon = \frac{1}{2} \frac{\rho(l_{\text{aero}})^3 s c_D}{I} \quad (\text{B-2})$$

and

$$\zeta^2 = \frac{m}{I} \frac{g l_{\text{c.g.}}}{I} \quad (\text{B-3})$$

Then equation (B-1) becomes

$$\ddot{\theta} + \epsilon \dot{\theta} |\dot{\theta}| + \zeta^2 \theta = 0 \quad 0 < \epsilon \ll 1 \quad (\text{B-4})$$

where the fact that  $0 < \epsilon \ll 1$  can be verified by the substitution of numerical values.

Extend the domain of the function to the additional variable

$$\tau = \epsilon^v g(t) \quad (B-5)$$

Now apply the principle of maximal balance to determine  $v$ :

$$\frac{d\theta}{dt} = \frac{\partial\theta}{\partial t} + \epsilon^v g' \frac{\partial\theta}{\partial \tau} \quad (B-6)$$

$$\frac{d^2\theta}{dt^2} = \frac{\partial^2\theta}{\partial t^2} + 2\epsilon^v g' \frac{\partial^2\theta}{\partial t\partial\tau} + \epsilon^{2v} g'^2 \frac{\partial^2\theta}{\partial \tau^2} + \epsilon^v g'' \frac{\partial\theta}{\partial \tau} \quad (B-7)$$

Then applying these derivatives back in the original differential equation results in

$$\begin{aligned} \frac{\partial^2\theta}{\partial t^2} + 2\epsilon^v g' \frac{\partial^2\theta}{\partial t\partial\tau} + \epsilon^{2v} g'^2 \frac{\partial^2\theta}{\partial \tau^2} + \epsilon^v g'' \frac{\partial\theta}{\partial \tau} + \epsilon \left( \frac{\partial\theta}{\partial t} \right)^2 + 2\epsilon^{v+1} g' \frac{\partial\theta}{\partial t} \frac{\partial\theta}{\partial \tau} \\ + \epsilon^{2v} g'^2 \left( \frac{\partial\theta}{\partial \tau} \right)^2 + \delta^2\theta = 0, \quad \frac{\partial\theta}{\partial t} > 0 \end{aligned} \quad (B-8)$$

or

$$\begin{aligned} \frac{\partial^2\theta}{\partial t^2} + 2\epsilon^v g' \frac{\partial^2\theta}{\partial t\partial\tau} + \epsilon^{2v} g'^2 \frac{\partial^2\theta}{\partial \tau^2} + \epsilon^v g'' \frac{\partial\theta}{\partial \tau} - \epsilon \left( \frac{\partial\theta}{\partial t} \right)^2 - 2\epsilon^{v+1} g' \frac{\partial\theta}{\partial t} \frac{\partial\theta}{\partial \tau} \\ - \epsilon^{2v+1} g'^2 \left( \frac{\partial\theta}{\partial \tau} \right)^2 + \delta^2\theta = 0, \quad \frac{\partial\theta}{\partial t} < 0 \end{aligned} \quad (B-9)$$

Consider the algebraic equations. They contain terms in the following powers of  $\epsilon$ :  $\epsilon^0$ ,  $\epsilon^v$ ,  $\epsilon^{2v}$ ,  $\epsilon$ ,  $\epsilon^{v+1}$ , and  $\epsilon^{2v+1}$ . Solving for  $v$  by equating exponents of various pairs of terms, the possible

values of  $v$  are:  $-1$ ,  $-\frac{1}{2}$ ,  $0$ ,  $\frac{1}{2}$ , and  $1$ . The  $\epsilon$  terms thus become, for each choice of  $v$ :

$v = 0$	$v = -1$	$v = -\frac{1}{2}$	$v = 1$	$v = +\frac{1}{2}$
$\epsilon^0$	$\epsilon^0$	$\epsilon^0$	$\epsilon^0$	$\epsilon^0$
$\epsilon^0$	$\epsilon^{-1}$	$\epsilon^{-\frac{1}{2}}$	$\epsilon$	$\epsilon^{-\frac{1}{2}}$
$\epsilon^0$	$\epsilon^{-2}$	$\epsilon^1$	$\epsilon^2$	$\epsilon^1$
$\epsilon^1$	$\epsilon$	$\epsilon$	$\epsilon$	$\epsilon^1$
$\epsilon^1$	$\epsilon^0$	$\epsilon^{\frac{1}{2}}$	$\epsilon^2$	$\epsilon^1$
$\epsilon^1$	$\epsilon^{-1}$	$\epsilon$	$\epsilon^3$	$\epsilon^{\frac{3}{2}}$

The best balance is yielded by  $v = 1$  (the case  $v = 0$ ) is the trivial case of no scaling at all). Thus (B-5) becomes

$$\tau = \epsilon g(t) \quad (B-10)$$

Now assume a solution of the form

$$\theta = \sum_{j=0}^{\infty} \theta_j(t, \tau) \epsilon^j \quad (B-11)$$

Then

$$\frac{d\theta}{dt} = \frac{\partial \theta}{\partial t} + \epsilon \dot{g} \frac{\partial \theta}{\partial \tau} = \theta_{0t} + \epsilon(\theta_{1t} + \dot{g} \theta_{0\tau}) + \epsilon^2(\theta_{2t} + \dot{g} \theta_{1\tau}) + \dots \quad (B-12)$$

$$\text{If } \dot{\theta} > 0, \epsilon \left| \frac{d\theta}{dt} \right| \frac{d\theta}{dt} = \epsilon(\theta_{0t})^2 + \epsilon^2(2\theta_{0t}(\theta_{1t} + \dot{g} \theta_{0\tau})) + \dots \quad (B-13)$$

$$\text{If } \dot{\theta} < 0, \epsilon \left| \frac{d\theta}{dt} \right| \frac{d\theta}{dt} = -\epsilon(\theta_{0t})^2 - \epsilon^2(2\theta_{0t}(\theta_{1t} + \dot{g} \theta_{0\tau})) \dots \quad (B-14)$$

$$\begin{aligned} \frac{d^2\theta}{dt^2} &= \frac{\partial^2\theta}{\partial t^2} + 2\epsilon \dot{g} \frac{\partial^2\theta}{\partial t \partial \tau} + \epsilon \ddot{g} \frac{\partial \theta}{\partial \tau} + \epsilon^2 \dot{g}^2 \frac{\partial^2\theta}{\partial \tau^2} = \theta_{0tt} + \epsilon(\theta_{1tt} + 2\dot{g}\theta_{0t\tau} + \ddot{g}\theta_{0\tau}) \\ &\quad + \epsilon^2(\theta_{2tt} + 2\dot{g}\theta_{1t\tau} + \ddot{g}\theta_{1\tau} + \dot{g}^2\theta_{0\tau\tau}) \end{aligned} \quad (B-15)$$

Thus if  $\dot{\theta} > 0$

$$\begin{aligned} &(\theta_{0tt} + \zeta^2\theta_0) + \epsilon[\theta_{1tt} + \zeta^2\theta_1 + 2\dot{g}\theta_{0t\tau} + \ddot{g}\theta_{0\tau} + \theta_{0t}^2] + \epsilon^2(\dots) + \epsilon^3(\dots) \\ &\quad + \dots = 0 \end{aligned} \quad (B-16)$$

The resultant system to second order is

$$\theta_{0tt} + \zeta^2\theta_0 = 0 \quad (B-17)$$

$$\theta_{1tt} + \zeta^2\theta_1 = -[2\dot{g}\theta_{0t\tau} + \ddot{g}\theta_{0\tau} + \theta_{0t}^2] \quad (B-18)$$

with the initial conditions at  $t = 0$

$$\theta_0 = 0 \quad , \quad \theta_{0t} = \dot{\theta}_0 \quad (B-19)$$

$$\theta_1 = 0 \quad , \quad \theta_{1t} + \dot{g}\theta_{0\tau} = 0$$

Also, if  $\dot{\theta} < 0$

$$\begin{aligned} (\theta_{0tt} + \zeta^2 \theta_0) + \epsilon(\theta_{1tt} + \zeta^2 \theta_1 + 2\dot{g}\theta_{0t\tau} + \ddot{g}\theta_{0\tau} - \theta_{0t}^2) + \epsilon^2(\dots) + \epsilon^3(\dots) \\ + \dots = 0 \end{aligned} \quad (B-20)$$

This system of equations is

$$\theta_{0tt} + \zeta^2 \theta_0 = 0 \quad (B-21)$$

$$\theta_{1tt} + \zeta^2 \theta_1 = - [2\dot{g}\theta_{0t\tau} + \ddot{g}\theta_{0\tau} - \theta_{0t}^2] \quad (B-22)$$

with the initial conditions at  $t=0$

$$\theta_0 = 0 \quad , \quad \theta_{0t} = \dot{\theta}_0 \quad (B-23)$$

$$\theta_1 = 0 \quad \theta_{1t} + \dot{g}\theta_{0\tau} = 0$$

The above two systems of equations are equivalent to



$$\theta_{0tt} + \zeta^2 \theta_0 = 0 \quad (B-24)$$

$$\theta_{1tt} + \zeta^2 \theta_1 = - [2\dot{g}\theta_{0t\tau} + \ddot{g}\theta_{0\tau} + |\theta_{0t}|\theta_{0t}] \quad (B-25)$$

with the initial conditions

$$\theta_0 = 0, \quad \theta_{0t} = \dot{\theta}_0 \quad (B-26)$$

$$\theta_1 = 0, \quad \theta_{1t} + \dot{g}\theta_{0t} = 0$$

Solving for  $\theta_0$

$$\theta_0 = A_0(\tau) \sin(\zeta t) \quad (B-27)$$

The function  $A_0$  is determined from the second equation:

$$\theta_{1tt} + \theta_1 = - [2\dot{g}\zeta A'_0(\tau) \cos(\zeta t) + \ddot{g}A'_0(\tau) \sin(\zeta t) + A_0^2(\tau) \zeta^2 \cos(\zeta t) |\cos(\zeta t)|] \quad (B-28)$$

In order for secular terms to be absent,

$$\int_0^{2\pi} \sin(\zeta t) \{ 2\dot{g}\zeta A'_0(\tau) \cos(\zeta t) + \ddot{g}A'_0(\tau) \sin(\zeta t) + A_0^2(\tau) \zeta^2 \cos(\zeta t) |\cos(\zeta t)| \} dt = 0 \quad (B-29)$$

and

$$\int_0^{2\pi} \cos(\zeta t) \{ 2\dot{g}\zeta A_0'(\tau) \cos(\zeta t) + \ddot{g}A_0'(\tau) \sin(\zeta t) + A_0^2(\tau) \zeta^2 \cos(\zeta t) |\cos(\zeta t)| \} dt = 0 \quad (B-30)$$

First, choose  $\ddot{g} = 0$

This implies  $\dot{g} = \zeta$  since the integration constant is arbitrary

$$g = \zeta t \quad (B-32)$$

and

$$\tau = \epsilon \zeta t \quad (B-33)$$

This simplifies the above equations to

$$\int_0^{2\pi} \sin(\zeta t) \{ 2\zeta^2 A_0'(\tau) \cos(\zeta t) + \zeta^2 A_0^2(\tau) \cos(\zeta t) |\cos(\zeta t)| \} dt = 0 \quad (B-34)$$

$$\int_0^{2\pi} \cos(\zeta t) \{ 2\zeta^2 A_0'(\tau) \cos(\zeta t) + \zeta^2 A_0^2(\tau) \cos(\zeta t) |\cos(\zeta t)| \} dt = 0 \quad (B-35)$$

The first equation is satisfied regardless of the value of  $A_0$ . The second equation yields

$$2\pi\zeta^2 A_0'(\tau) + \frac{8}{3} \zeta^2 A_0^2(\tau) = 0 \quad (\text{B-36})$$

The frequency term drops out. Hence

$$\frac{A_0'(\tau)}{A_0^2(\tau)} = -\frac{4}{3\pi} \quad (\text{B-37})$$

Thus

$$\frac{1}{A_0(\tau)} = \frac{4\tau}{3\pi} + K \quad (\text{B-38})$$

$$= \frac{4\epsilon\zeta t}{3\pi} + K$$

or

$$A_0(\tau) = \frac{1}{\frac{4\epsilon\zeta t}{3\pi} + K} \quad (\text{B-39})$$

It is now necessary to solve for K. From (B-27), taking the time derivative

$$\frac{\partial \theta_0}{\partial t} = \delta A_0(\tau) \cos(\zeta t) + \frac{\partial A_0(\tau)}{\partial t} \sin(\zeta t) \quad (\text{B-40})$$

If this is evaluated at  $t = 0$

$$\left. \frac{\partial \theta_0}{\partial t} \right|_{t=0} = \zeta A_0(0) \quad (B-41)$$

But from the initial conditions (B-26) this is equal to  $\dot{\theta}_0$ , so

$$A_0(0) = \frac{\dot{\theta}_0}{\zeta} \quad (B-42)$$

also from (B-39) at  $t = 0$

$$A_0(0) = \frac{1}{K} \quad (B-43)$$

Hence

$$K = \frac{\zeta}{\dot{\theta}_0} \quad (B-44)$$

So, in (B-39)

$$A_0(\tau) = \frac{\dot{\theta}_0}{\zeta} \left[ \frac{1}{1 + \left( \frac{4\epsilon \dot{\theta}_0}{3\pi} t \right)} \right] \quad (B-45)$$

The final solution is then

$$\theta_0 = \frac{\dot{\theta}_0}{\zeta} \left[ \frac{1}{1 + \left( \frac{4\epsilon \dot{\theta}_0}{3\pi} t \right)} \right] \sin(\zeta t) + o(\epsilon) \quad (B-46)$$

It can be seen that the amplitude is strongly influenced by the frequency. However, the period remains essentially the same as for

undamped motion. This solution is essentially the same as that of Bogoliubov and Mitropolsky (ref. 15), p. 75-77. However, it is thought that the method used here is somewhat simpler and more elegant.

## VITA

The author was born in [REDACTED] on [REDACTED]  
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